STELLA MARY'S COLLEGE OF ENGINEERING

(Accredited by NAAC, Approved by AICTE - New Delhi, Affiliated to Anna University Chennai)

Aruthenganvilai, Azhikal Post, Kanyalumari District, Tamilnadu - 629202.

ME8692 FINITE ELEMENT ANALYSIS

(Anna University: R2017)

Prepared By

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Aruthenganvilai, Kallukatti Junction Azhikal Post, Kanyakumari District-629202,
Tamil Nadu.

DEPARTMENT OF MECHANICAL ENGINEERING

COURSE MATERIAL

REGULATION	**2017**
YEAR	**III**
SEMESTER	**06**
COURSE NAME	**FINITE ELEMENT ANALYSIS**
COURSE CODE	**ME8692**
NAME OF THE COURSE INSTRUCTOR	**Mr. S. AJITH KUMAR**

SYLLABUS:

UNIT I INTRODUCTION 9

Historical Background – Mathematical Modeling of field problems in Engineering – Governing Equations – Discrete and continuous models – Boundary, Initial and Eigen Value problems–Weighted Residual Methods – Variational Formulation of Boundary Value Problems – Ritz Technique – Basic concepts of the Finite Element Method.

UNIT II ONE-DIMENSIONAL PROBLEMS 9

One Dimensional Second Order Equations – Discretization – Element types- Linear and Higher order Elements – Derivation of Shape functions and Stiffness matrices and force vectors- Assembly of Matrices - Solution of problems from solid mechanics and heat transfer. Longitudinal vibration frequencies and mode shapes. Fourth Order Beam Equation –Transverse deflections and Natural frequencies of beams.

UNIT III TWO DIMENSIONAL SCALAR VARIABLE PROBLEMS 9

Second Order 2D Equations involving Scalar Variable Functions – Variational formulation –Finite Element formulation – Triangular elements – Shape functions and element matrices and vectors. Application to Field Problems - Thermal problems – Torsion of Non circular shafts –Quadrilateral elements – Higher Order Elements.

UNIT IV TWO DIMENSIONAL VECTOR VARIABLE PROBLEMS 9

Equations of elasticity – Plane stress, plane strain and axisymmetric problems – Body forces and temperature effects – Stress calculations - Plate and shell elements.

UNIT V ISOPARAMETRIC FORMULATION 9

Natural co-ordinate systems – Isoparametric elements – Shape functions for iso parametric elements – One and two dimensions – Serendipity elements – Numerical integration and application to plane stress problems - Matrix solution techniques – Solutions Techniques to Dynamic problems – Introduction to Analysis Software.

TEXT BOOKS:

1. Reddy. J.N., “An Introduction to the Finite Element Method”, 3rd Edition, Tata McGraw-Hill, 2005.
2. Seshu, P, “Text Book of Finite Element Analysis”, Prentice-Hall of India Pvt. Ltd., New Delhi, 2007.

REFERENCES:

1. Bhatti Asghar M, "Fundamental Finite Element Analysis and Applications", John Wiley & Sons, 2005 (Indian Reprint 2013)*.
2. Chandrupatla & Belagundu, “Introduction to Finite Elements in Engineering”, 3rd Edition, Prentice Hall College Div, 1990.
3. Logan, D.L., “A first course in Finite Element Method”, Thomson Asia Pvt. Ltd., 2002
4. Rao, S.S., “The Finite Element Method in Engineering”, 3rd Edition, Butterworth Heinemann,2004.
5. Robert D. Cook, David S. Malkus, Michael E. Plesha, Robert J. Witt, “Concepts and Applications of Finite Element Analysis”, 4th Edition, Wiley Student Edition, 2002.

Course Outcome Articulation Matrix

	Program Outcome												***PSO***		
Course Code / CO No	***1***	***2***	***3***	***4***	***5***	***6***	***7***	***8***	***9***	***10***	***11***	***12***	***1***	***2***	***3***
ME8693 / C319.1	3	3	2	3	0	0	0	0	1	0	0	3	3	1	3
ME8693 / C319.2	3	3	2	3	0	0	0	0	1	0	0	3	3	1	3
ME8693 / C319.3	3	3	2	3	0	0	0	0	1	0	0	3	3	1	3
ME8693 / C319.4	3	3	2	3	0	0	0	0	1	0	0	3	3	1	3
ME8693 / C319.5	3	3	2	3	0	0	0	0	1	0	0	3	3	1	3
Average	3	3	2	3	0	0	0	0	1	0	0	3	3	1	3

FINITE ELEMENT ANALYSIS.

UNIT - 1 : INTRODUCTION.

Historical Background - Mathematical modeling of field problems in engg - Governing Equations - Discrete & continuous models - Boundary, initial & Eigen value Problems - weighted Residual methods - variational formulation of boundary value problems - Ritz Technique - Basic concepts of the finite element method.

Fundamental concepts of Engg. Analysis:-

✓ Analyst needs certain requirements while designing & assembling the parts of the product.

✓ To calculate displacement, shear, Pressure, temp etc.

Methods of Engg. Analysis.

1) Experimental Methods - Prototype can be used to find the various requirements

2) Theoretical (or) Analytical Methods - Mathematical differential equation

3) Numerical Methods - Approximate method but gives acceptable solutions

a) Functional Approximation

b) Finite Difference Method (FDM)

c) Finite Element Method (FEM) / FEA

Functional Approximation -

✓ Physical problem is written in mathematical expression and by integrating + imposing boundary conditions, the solution is det.

✓ Ex: Rayleigh Ritz method;
weighted residual method.

Finite Differential method:

✓ For solving heat transfer fluid mechanics + structural mechanics problems

✓ Applied to any phenomenon for which differential equation along with boundary conditions are available.

✓ Difficult to use in curved (or) irregular boundaries + in computer programmed.

Finite Element Method (FEM) / Finite Element Analysis (FEA)

✓ Numerical method for solving Engg and mathematical physics.

✓ A structure / Body is subdivided into smaller elements of finite dimensions (called elements) and assemble these elements and solve to get the whole solution.

✓ It is used in solving physical problems involving complicated geometries, loading + materials properties

✓ Based on application the finite element problems are classified as,

structural - displacement, stress & strain etc.

Non- structural - Temp (heat flow) fluid flow etc.

Historical Background of FEM:

✓ Basic ideas of finite element analysis were developed by aircraft engineers in early 1940's.

✓ Morden development began in the year 1945 in the structural field by Hei Hrennikoff.

✓ In 1947 Levy introduced the flexibility method & in 1953 he suggested stiffness method for use in analysing statically redundant aircraft structures.

✓ In 1954, Argyris & Kelsey developed matrix structural analysis method using energy Principles.

✓ In 1960, Clough introduced finite element in the plane stress analysis and he used triangular & rectangular elements.

✓ In 1961, Turner considered large deflection and thermal analysis Problems.

✓ In 1960, Gallagher introduced material non-linearity Problems.

✓ In 1968, Zinkiewicz extended the method to visco- elasticity Problems.

✓ In 1967, Szabo & lee introduced weighted residual method & In 1970 zinkiewicz & Pareth for transient field problems.

✓ In the decades of 1960's & 1970's, the finite element method was extended to applications in.

- ✓ Shell bending.
- ✓ Plate bending.
- ✓ Heat transfer Analysis.
- ✓ Fluid flow Analysis &
- ✓ 3Dimensional Problems in structural Analysis.

Basics of FEM:

✓ Finite Element method is an integral Part of computer aided Engineering (CAE) and is being extensively used in the analysis and design of many complex real-life systems.

✓ FEM can be adopted in which the whole body or system is divided into many small elements whose characteristics are studied and then rebuilt the original system to understand the entire system behaviour.

✓ It's a numerical procedure for solving physical Problems governed by a differential equation (or) an energy theorem.

✓ Although, FEM solutions is not exact, the solution can be improved by using more elements to represent the physical problems.

Advantages:-

- can model irregular shaped bodies quite easily
- ✓ can handle general load conditions without any difficulty.
- ✓ can handle different kinds of boundary conditions.
- ✓ Includes dynamic effects.
- model bodies composed of several different materials. Element equations are evaluated individually.
- ✓ Handles non-linear behaviour existing with large deformations. & Non-linear materials.
- Non-homogeneous materials can be handled easily.
- ✓ Higher order elements may be implemented.

Limitations:

- ✓ It's a time consuming Process
- ✓ Results obtained from FEM will be closer to exact solution. only if the system is devide into more smaller elements.
- ✓ It requires a digital computer and higher-end softwares.

Applications of FEA / FEM:

- * Mechanical design
  - → stress analysis of pressure vessels, pistons, composite materials, linkages etc.
  - → Temperature distribution in solids & fluids
  - → Analysis of potential flow, free surface flows, viscous flows.

- * civil Engineering.
  - → Analysis of trusses, frames, bridges. hydraulic structures & dams.

- * Aircraft structures:
  - → Aircraft wings, fins, Rockets, space crafts.

- * Electrical Machines:
  - → Analysis of synchronous and induction machines, eddy current & core losses in electric machines.

- * Geo-mechanics
  - → stress analysis in solids, dams, machine foundation.

- * Bio-medical Engineering.
  - → stress Analysis of Eye balls, bones & teeths, mechanics of heart valves.

FEA software:

- * NASTRAN
- * DYNA
- * I - DEAS
- * ANSYS
- * ABAQUS
- * SAP
- * ASKA
- * COSMOS
- * ADINA.

Procedure / steps in FEA:

1. Pre processing
2. Analysis
3. Post processing.

Two general methods are associated with FEA.

* Force method - Internal forces are considered as the unknowns.

* Displacement (or) stiffness method - Displacements of the nodes are considered as unknowns.

* Among the 2 methods: Displacement method is more desirable because of simple formulation.

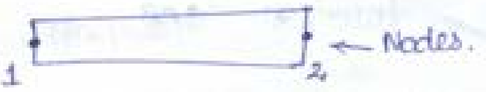
① Pre processing: i) Discretization
2) Numbering of Nodes and Elements.

1. Discretization:

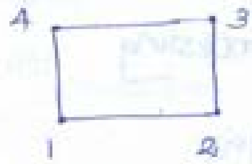
Art of sub dividing a structure into a convenient number of smaller elements.

smaller elements are classified as.

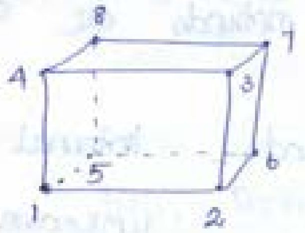
* One- dimensional Elements : Bar, Beam, Truss.



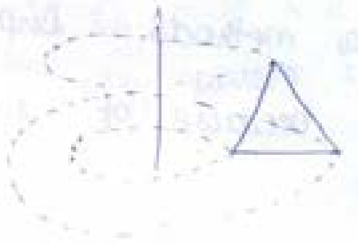
* Two dimensional Elements : Triangle. Rectangle.



* Three dimensional Element : Tetrahedral Element ; Hexahedral (or) Brick Element



* Axisymmetric Element : Rotating triangle (or) Quadrilateral about a fixed axis.

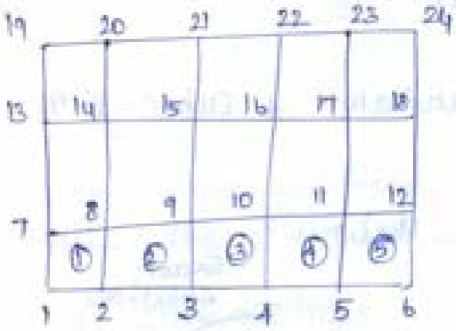


2) Numbering of Nodes & Elements :

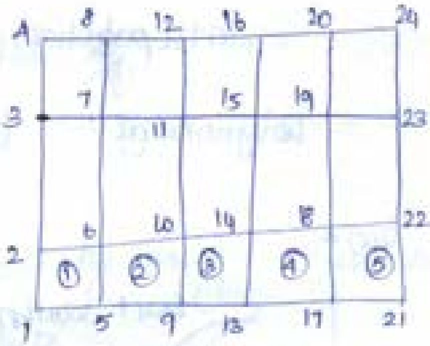
NO. Process is important, it decides the size of the stiffness matrix.

$$\left\{\begin{matrix}\text{Maximum node}\\\text{number}\end{matrix}\right\} - \left\{\begin{matrix}\text{Minimum}\\\text{Node number}\end{matrix}\right\} = \text{Minimum}$$

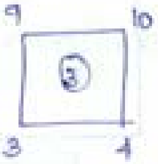
Longer side Numbering

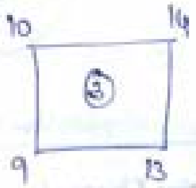


Shorter side Numbering



Element 3 :





(Max. Node) – (Min. Node) = Min.

10 - 3 = 7 ;

14 - 9 = 5

✓ shorter side Numbering is followed in FEA

II Analysis

3. Selection of displacement function.

4. Define the material behaviour.

5. Derivation of Element stiffness matrix and Equ.

6. Assemble the element Equations.

7. Applying Boundary conditions.

8. Solution of unknown displacements.

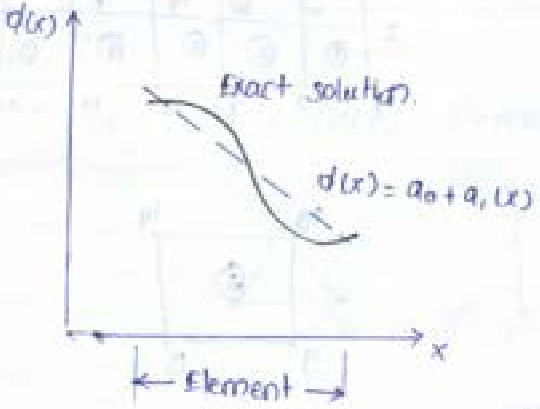
9. computation of the element stress and strain.

III Post Processing

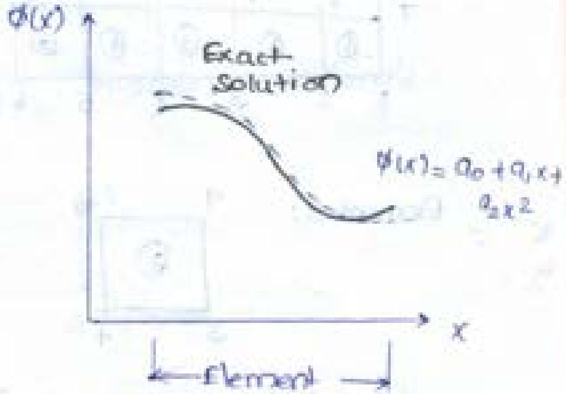
10. Interpret the results.

3) Selection of a Displacement function (or) Interpolation function.

Polynominal : linear, quadratic & cubic form.



a) Linear Approximation



b) Quadratic Approximation.

Polynomial type are mostly use due to,

- ✓ Easy formulation & computerize the finite Element Equation.
- ✓ Easy to perform differentiation (or) Integration.
- ✓ Accuracy can be improved by increasing the polynominal order.

linear Polynominal:

1D problem $\phi(x) = a_0 + a_1 x$

2D problem: $\phi(x, y) = a_0 + a_1 x + a_2 y$

3D problem $\phi(x, y, z) = a_0 + a_1 x + a_2 y + a_3 z$.

Quadratic polynominal:

1D : $\phi(x) = a_0 + a_1 x + a_2 x^2$

2D $\phi(x,y) = a_0 + a_1 x + a_2 y + a_3 x^2 + a_4 y^2 + a_5 xy$

3D $\phi(x,y,z) = a_0 + a_1 x + a_2 y + a_3 z + a_4 x^2 + a_5 y^2$
$+ a_6 z^2 + a_7 xy + a_8 yz + a_9 xz.$

4) Define the material behaviour by using stress-Displac. & stress-strain Relationships.

For one-dimensional deformation.

$$e = \frac{du}{dx}$$

u → displacement variable in x direction
e → strain.

stress-strain Relationship.

$$\sigma = E \cdot e$$

σ → stress in x direction
E → Youngs modulus (or) modulus of Elasticity.

5) Derivation of Element stiffness matrix & Equations.

finite Element Equation.

$$\{F^e\} = [K^e][u^e]$$

e → Element ; {F} vector of Element nodal forces.
{K} stiffness matrix of the Element
{u} Displacement vector

In matrix form.

$$\begin{Bmatrix} f_1 \\ f_2 \\ f_3 \\ \vdots \\ f_n \end{Bmatrix} = \begin{bmatrix} K_{11} & K_{12} & K_{13} & \cdots & K_{1n} \\ K_{21} & K_{22} & K_{23} & \cdots & K_{2n} \\ K_{31} & K_{32} & K_{33} & \cdots & K_{3n} \\ & & \vdots & & \\ K_{n1} & K_{n2} & K_{n3} & & K_{nn} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_n \end{Bmatrix}$$

This Equation can be derived by,

i) Direct Equilibrium method - Line (or) One dimensional Element

2) Variational method - complex Elements.

Ex: Axisymmetric stress element, Plate bending Element & 2 (or) 3 D solid stress Element

3) Weighted Residual method (or) Galerkin's method - Thermal Analysis Problems.

✓ used when potential energy is not readily available.

6) Assemble the Elemental Equation to obtain global Equation.

$$\underset{\text{Global force vector}}{\{F\}} = \underset{\text{Global stiffness matrix}}{[K]} \underset{\text{Global displacement Vector}}{\{u\}}$$

7) Applying Boundary conditions:

2 types: ① Geometric (or) Essential ② Natural (or) Non-Essential (or) Force boundary condition.

8) Solution for the unknown displacements.

✓ unknown displacements {u} are calculated by using Gaussian Elimination method (or) Gauss - Seidel method.

9) Computation of the Element strains & stresses from the nodal displacements {u}:

In one-dimensional formation,

$$\text{strain, } e = \frac{du}{dx} = \frac{u_2 - u_1}{x_2 - x_1}$$

$$\text{strain, } \sigma = E \cdot e$$

Discretization:-

✓ The act of subdividing a structure into a convenient no. of smaller components (Elements).

✓ smaller components are put together is called Assemblage

Characteristics of an Element.

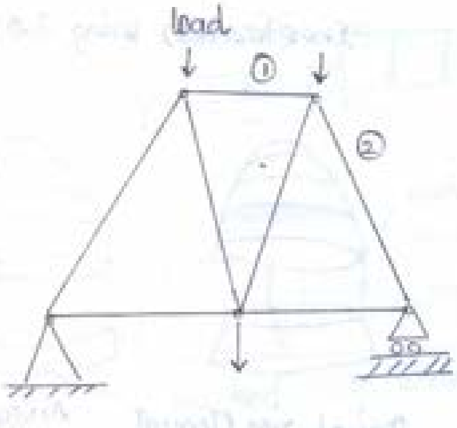
✓ It is a small portion of a system.

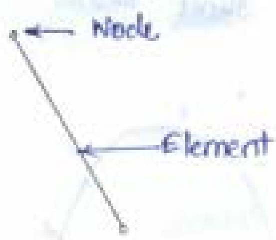
✓ It has definite shape

✓ It should have min. of 2 Nodes.

✓ Nodes are placed where connection is made to another Element

✓ Loads act only at the nodes.



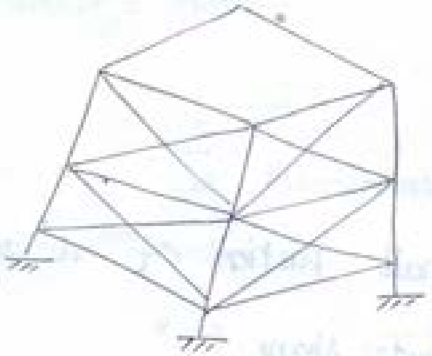


Discretization can be classified as,

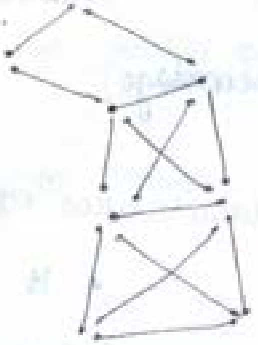
- ✓ Natural
  - Ex: Truss.
- ✓ Artificial (continuum)
  - Triangular Element
  - Rectangular "
  - Quadrilateral "

Discretization Process :

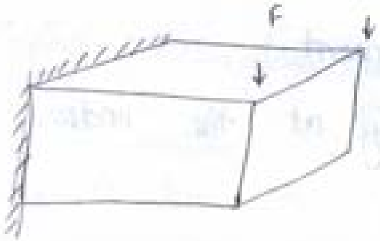
(i) Type of Element :



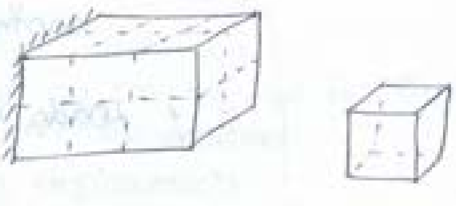
Original Structure



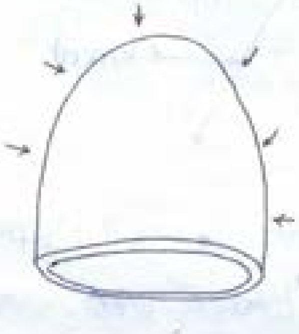
Discretization using bar Element



Short Beam

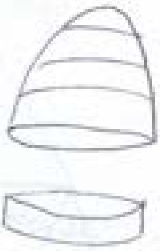


Discretization using 3.D Element



Original Shell

Conical ring Element



Axisymmetric Element

Element

choice of the Element, based on

- ✓ No. of degrees of freedom
- ✓ Expected Accuracy.
- ✓ Necessary Equations required.

ii) Size of Element :-

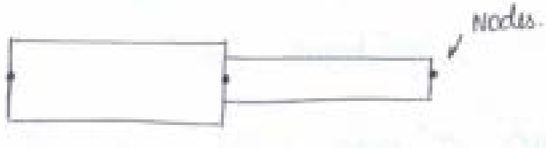
Size of Element is small → final solution is more accurate.

Aspect ratio → Ratio of the largest dimension to the smallest dimensions of the Element.

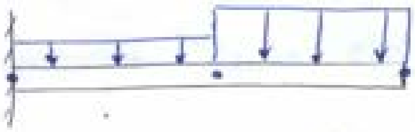
iii) Locations of Nodes:

✓ If the structure has no abrupt change in geometric, load, boundary conditions & material properties, the structure can be divided into equal subdivisions.

✓ If any discontinuities in geometric, load, boundary condition & material properties, nodes should be introduced at these discontinuities in the structure.



Geometric discontinuities



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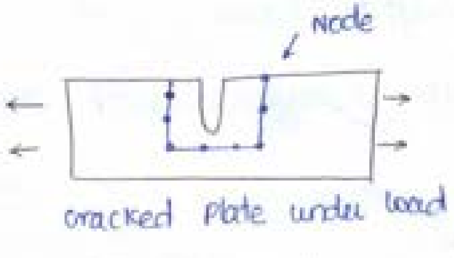
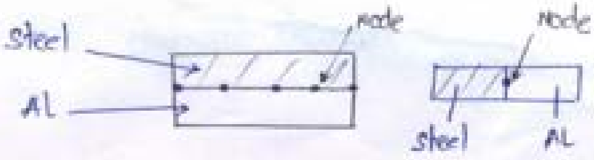


Boundary condition "



Material discontinuity.

The functional approximation methods for solving the different boundary conditions problems are classified as,

1. Variational method.
2. weighted residual method.

i) Variational method:

Most of continuum Problem (whole system representing Problem) can be specified in 2 ways.

① Problems are specified in suitable differential equations governing the behaviour along with the boundary conditions.

② Physical Problem expressed by a differential equation can be represented interms of equivalent integral form.

In variational method,

✓ Physical problem expressed in terms of differential equation is recast in an equivalent integral form.

✓ with the help of some trial functions. (functional) is made to reach maximum (or) minimum conditions.

ii) Weighted residual method:

✓ It is employed to obtain approximate solutions to linear & Non-linear Problems interms of differential equations.

Vaiational Method:

The physical problem expressed interms of differential equation is recast in an equivalent integral fom. Then, with the help of some trial function called functional is made to reach the extreme conditions.

In Rayleigh ritz method, the Principle of variational approach is used to find the solution

Rayleigh - Ritz method:

Rayleigh - Ritz method in a typical variational method in which the principle of integral approach is adepted for solving mostly the complex structural problems.

In this method, Problems are solved in 2 ways.

1. Minimum potential energy Method
2. Integral approach Method.

Ritz method based on Potential energy concept.

Total potential energy π is considered as the function of generalised co-ordinates which are exactly equal to the number of degrees of freedom.

In Rayleigh - Ritz method, the total potential energy π is considered as the function of Ritz Parameters.

To solve a Problem in Ritz method,

- displacement function is assumed interms of Ritz co-efficients.
- displacement function $y(x)$ is expressed in terms of Polynomial series (or) Trignometric series.

$$y(x) = a_1 + a_2 x + a_3 x^2 + \cdots \quad \text{(or)}$$

$$y(x) = a_1 \sin \frac{\pi x}{l} + a_2 \sin \frac{3\pi x}{l} + \cdots$$

$a_1, a_2 \cdots$ Ritz co-efficients / Parameters.

✓ Based on the displacement function, the total Potential energy can be formulated.

Total potential energy, $\pi = U - W$

U- Internal strain energy

W- workdone by the external force

Ritz method based on Integral Approach:

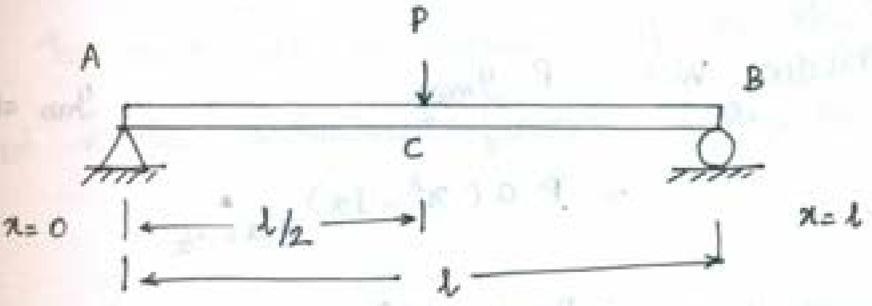
Integral of the trial function (functional) is used to find the approximate solution by the differential equation.

$$\text{Function, } I = \int_D y(x)\, dx$$

$$\frac{\partial I}{\partial a} = 0$$

Problems based on Potential energy Approach:

Find the deflection at the centre of a simply supported beam of span length L is subjected to a concentrated load P at its mid-point as shown in fig. use Rayleigh-Ritz method.



Solution,

The total potential energy, $\pi = U - W$

Strain energy for a beam, $U = \frac{EI}{2}\int_0^L \left(\frac{d^2y}{dx^2}\right)^2 . dx$

$y(x)$: displacement function, expressed as

$y(x) = a_1 + a_2 x + a_3 x^2$ — (1) (only 3 terms for simplification)

Boundary conditions: at $x = 0$; $y = 0$ and
$x = l$

Hence, Eqn (1) ⇒ $0 = a_1$

$0 = a_2 l + a_3 l^2$; $a_2 = -a_3 l$

$\therefore y(x) = 0 + (-a_3 l)x + a_3 x^2$

$= a_3 (x^2 - lx)$

Let $a_3 = a$

$y(x) = a(x^2 - lx)$ (or) — (3)

$ax(x - l)$

$\frac{dy}{dx} = a(2x-l)$; $\frac{d^2y}{dx2} = 2a$

∴ Total strain energy, $U = \frac{EI}{2}\int_0^L (2a)^2 . dx$

$$= \frac{EI}{2}(4a^2)\,[x]_0^l$$

$$= 2EIa^2 l$$

workdone $W = P.\, y_{max}$ y_{max} at $x = l/2$

$$= P\,a(x^2 - lx)_{@\, x = l/2}$$

$$= Pa\,((l/2)^2 - lx)$$

$$W = -Pa\, l^2/4$$

∴ Total Potential energy, $\pi = U - W$

$$\pi = 2EIa^2 l + Pa\, l^2/4$$

For minimum potential energy condition, $\frac{d\pi}{da} = 0$

$$\frac{\partial \pi}{\partial a} = 4EIal + Pl^2/4 = 0$$

$$a = \frac{-Pl}{16EI}$$

Substitute in Eq(3), $y(x) = \frac{-PL}{16EI}(x^2 - lx)$

maximum deflection at $x = l/2$

$$y_{max} = \frac{-Pl}{16EI}\left[\frac{l^2}{4} - l.\frac{l}{2}\right]$$

$$y_{max} = \frac{Pl^3}{64EI}$$ (Approximate solution).

Exact solution: $$y_{max} = \frac{Pl^3}{48EI}$$

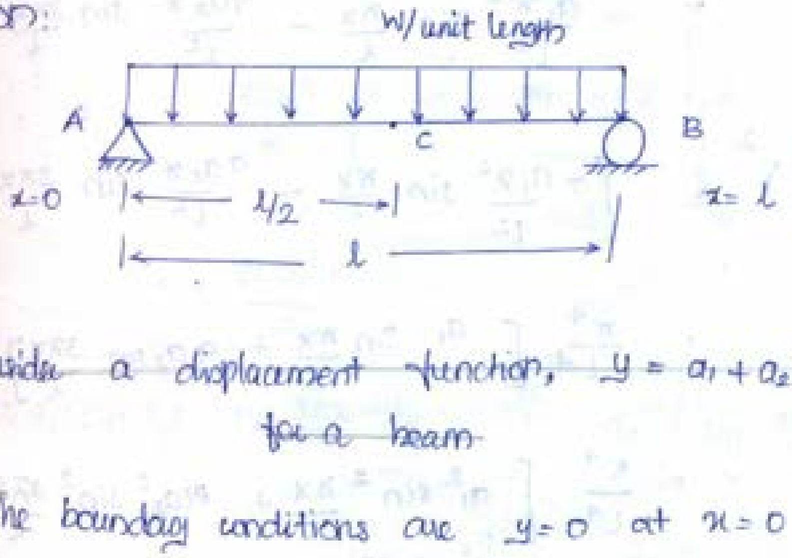
The result will be accurate, if the Ritz coefficient added in the displacement function (a_4, $a_5 \cdots$)

-.

Derive the expression for deflection and bending moment in a simply supported beam of span length l, subjected to uniformly distributed load over entire span using two term trigonometric trial function.

Also find the deflection and moment at mid-span and compare with exact solution. use Ritz method.

Solution:



~~Consider a displacement function, $y = a_1 + a_2 x + a_3 x^2$ for a beam~~

The boundary conditions are $y = 0$ at $x = 0$ and $x = L$

deflection for a beam interms of two term trigonometric trial function.

$$y = a_1 \sin\frac{\pi x}{L} + a_2 \sin\frac{3\pi x}{l}$$

Since, the above selected trial function satisfies the boundary conditions. This function is correct

Now, applying the total potential energy concept,

$$\pi = U - W$$

strain energy for a beam,

$$U = \frac{EI}{2}\int_0^l \left(\frac{d^2y}{dx^2}\right)^2 dx$$

$$\frac{dy}{dx} = a_1\left(\cos\frac{\pi x}{l}\right)\frac{\pi}{l} + a_2\left(\cos\frac{3\pi x}{l}\right)\frac{3\pi}{l}$$

$$= \frac{a_1\pi}{l}\cos\frac{\pi x}{l} + \frac{3a_2\pi}{l}\cos\frac{3\pi x}{l}$$

$$\frac{d^2y}{dx^2} = \frac{a_1\pi}{l}\left(-\sin\frac{\pi x}{l}\right)\left(\frac{\pi x}{l}\right) + \frac{3a_2\pi}{l}\left(-\sin\frac{3\pi x}{l}\right)\left(\frac{3\pi}{l}\right)$$

$$= -\frac{a_1\pi^2}{L^2}\sin\frac{\pi x}{l} - \frac{9a_2\pi^2}{l^2}\sin\frac{3\pi x}{l}$$

$$\therefore \left(\frac{d^2y}{dx}\right)^2 = \left[-\frac{a_1\pi^2}{l^2}\sin\frac{\pi x}{l} - \frac{9a_2\pi^2}{l^2}\sin\frac{3\pi x}{l}\right]^2$$

$$= \frac{\pi^4}{L^4}\left[a_1\sin\frac{\pi x}{l} + 9a_2\sin\frac{3\pi x}{l}\right]^2$$

$$= \frac{\pi^4}{L^4}\left[a_1^2\sin^2\frac{\pi x}{l} + 81a_2^2\sin^2\frac{3\pi x}{l} + 18a_1a_2\sin\frac{\pi x}{l}\sin\frac{3\pi x}{l}\right]$$

∴ Strain energy,

$$U = \frac{EI}{2} \int_0^l \frac{\pi^4}{l^4} \left(a_1^2 \sin^2 \frac{\pi x}{l} + 81 a_2^2 \sin^2 \frac{3\pi x}{l} + 18 a_1 a_2 \sin \frac{\pi x}{l} \sin \frac{3\pi x}{l} \right) dx$$

$$= \frac{EI\pi^4}{2l^4} \int_0^l \left(a_1^2 \sin^2 \frac{\pi x}{l} + 81 a_2^2 \sin^2 \frac{3\pi x}{l} + 18 a_1 a_2 \sin \frac{\pi x}{l} \sin \frac{3\pi x}{l} \right) dx$$

Integrate the above equation individually,

$$\int_0^L a_1^2 \sin^2 \frac{\pi x}{l}\, dx = a_1^2 \int_0^L \frac{1}{2} \left(1 - \cos \frac{2\pi x}{l}\right) dx$$

$$= \frac{a_1^2}{2} \left[x - \frac{\sin \frac{2\pi x}{l}}{\frac{2\pi}{l}} \right]_0^l$$

$\sin^2 A = \frac{1}{2}(1 - \cos 2A)$

$$= \frac{a_1^2}{2} \left[(l - 0) - \frac{L}{2\pi} (\sin 2\pi - \sin 0) \right]$$

$$= \frac{a_1^2}{2} \left[(l - 0) - \frac{L}{2\pi} (0 - 0) \right]$$

$\sin 0 = 0$

$\sin 2\pi = 0$

$$= \frac{a_1^2 L}{2}$$

$$\int_0^L 81 a_2^2 \sin^2 \frac{3\pi x}{l} . dx = 81 a_2^2 \int_0^l \frac{1}{2} \left(1 - \cos \frac{6\pi x}{l}\right) dx$$

$$= \frac{81 a_2^2}{2} \left[x - \frac{\sin \frac{6\pi x}{l}}{\frac{6\pi}{l}} \right]_0^l$$

$$= \frac{81 a_2^2 l}{2}$$

$$\int_0^L 18 a_1 a_2 \sin \frac{\pi x}{l} . \sin \frac{3\pi x}{l} . dx = 18 a_1 a_2 \int_0^l \sin \frac{3\pi x}{l} . \sin \frac{\pi x}{l}$$

$$= 18 a_1 a_2 \int_0^l \frac{1}{2} \left(\cos \frac{2\pi x}{l} - \cos \frac{4\pi x}{l} \right) . dx$$

*

$\sin A . \sin B = \frac{1}{2} [\cos(A-B) - \cos(A+B)]$

$$= 9a_1a_2 \left[\frac{\sin \frac{2\pi x}{l}}{\left(\frac{2\pi}{l}\right)} - \frac{\sin \frac{4\pi x}{l}}{\left(\frac{4\pi}{l}\right)} \right]_0^l$$

$$= 9a_1a_2 \left[\frac{l}{2\pi} (\sin 2\pi - \sin 0) - \frac{l}{4\pi} (\sin 4\pi - \sin 0) \right]$$

$$= 9a_1a_2 \left[\frac{l}{2\pi} (0-0) - \frac{l}{4\pi} (0-0) \right]$$

$$= 0.$$

Substituting, we get

$$U = \frac{EI\pi^4}{2l^4} \left[\frac{a_1^2 l}{2} + \frac{81 a_2^2 l}{2} \right].$$

$$= \frac{EI\pi^4}{4l^3} \left[a_1^2 + 81a_2^2 \right]$$

workdone by the external force

$$W = \int_0^l w . y . dx$$

$$= \int_0^l w \left(a_1 \sin \frac{\pi x}{l} + a_2 \sin \frac{3\pi x}{l} \right) dx$$

$$= w \left[a_1 \frac{-\cos \frac{\pi x}{l}}{\frac{\pi}{l}} \right]_0^l + \left[a_2 \frac{-\cos \frac{3\pi x}{l}}{\frac{3\pi}{l}} \right]_0^l$$

$$= w \left[-\frac{a_1 l}{\pi} (\cos \pi - \cos 0) - \frac{a_2 l}{3\pi} (\cos 3\pi - \cos 0) \right]$$

$$= w \left[-\frac{a_1 l}{\pi} (-1-1) - \frac{a_2 l}{3\pi} (-1-1) \right]$$

$$= w \left[\frac{2a_1 l}{\pi} + \frac{2a_2 l}{3\pi} \right]$$

$$= \frac{2wl}{\pi} \left(a_1 + \frac{a_2}{3} \right)$$

$\cos \pi = -1$

$\cos 3\pi = -1$

$\cos 0 = 1$

Total potential energy, $\pi = U - W$

$$= \frac{EI}{4l^3}\pi^4\left[a_1^2 + 81a_2^2\right] - \frac{2wL}{\pi}\left[a_1 + \frac{a_2}{3}\right]$$

for minimum potential energy,

$$\frac{\partial\pi}{\partial a_1} = 0; \quad \frac{\partial\pi}{\partial a_2} = 0;$$

$$\frac{\partial\pi}{\partial a_1} = \frac{EI\pi^4}{4l^3}(2a_1) - \frac{2wl}{\pi} = 0; \quad a_1 = \frac{4wL^4}{EI\pi^5}$$

$$\frac{\partial\pi}{\partial a_2} = \frac{EI\pi^4}{4l^3}(162a_2) - \frac{2wL}{3\pi} = 0; \quad a_2 = \frac{4wL^4}{243\,EI\pi^5}$$

Hence, the deflection,

$$y = a_1 \sin\frac{\pi x}{l} + a_2 \sin\frac{3\pi x}{l}$$

$$= \frac{4wL^4}{EI\pi^5}\sin\frac{\pi x}{l} + \frac{4wL^4}{243EI\pi^5}\sin\frac{3\pi x}{l}$$

Maximum deflection at $x = l/2$,

$$\therefore y_{max} = \frac{4wL^4}{EI\pi^5}\sin\frac{\pi(l/2)}{l} + \frac{4wL^4}{243EI\pi^5}\sin\frac{3\pi(l/2)}{l}$$

$$= \frac{4wL^4}{EI\pi^5}\left[1 - \frac{1}{243}\right]$$

$\sin\frac{\pi}{2} = 1$

$\sin\frac{3\pi}{2} = -1$

$$y_{max.} = \frac{wL^4}{76.81\,EI}$$

Exact solution of maximum deflection of beam,

$$y_{max} = \frac{5}{384}\frac{wL^4}{EI} = \frac{wL^4}{76.8\,EI}$$

(Almost equal to the exact solution).

Total potential energy, $\pi = U - W$

$$= \frac{EI}{4l^3} \pi^4 \left[a_1^2 + 81 a_2^2 \right] - \frac{2wL}{\pi} \left[a_1 + \frac{a_2}{3} \right]$$

∴ minimum potential energy,

$$\frac{\partial \pi}{\partial a_1} = 0; \quad \frac{\partial \pi}{\partial a_2} = 0;$$

$$\frac{\partial \pi}{\partial a_1} = \frac{EI\pi^4}{4l^3}(2a_1) - \frac{2wl}{\pi} = 0; \quad a_1 = \frac{4wL^4}{EI\pi^5}$$

$$\frac{\partial \pi}{\partial a_2} = \frac{EI\pi^4}{4l^3}(162a_2) - \frac{2wl}{3\pi} = 0; \quad a_2 = \frac{4wL^4}{243 EI\pi^5}$$

Hence, the deflection,

$$y = a_1 \sin\frac{\pi x}{l} + a_2 \sin\frac{3\pi x}{l}$$

$$= \frac{4wL^4}{EI\pi^5} \sin\frac{\pi x}{l} + \frac{4wL^4}{243EI\pi^5} \sin\frac{3\pi x}{l}$$

Maximum deflection at $x = l/2$.

$$\therefore y_{max} = \frac{4wL^4}{EI\pi^5} \sin\frac{\pi(l/2)}{l} + \frac{4wL^4}{243EI\pi^5} \sin\frac{3\pi(l/2)}{l}$$

$$= \frac{4wL^4}{EI\pi^5}\left[1 - \frac{1}{243}\right] \qquad \left|\ \sin\frac{\pi}{2} = 1, \quad \sin\frac{3\pi}{2} = -1\right.$$

$$y_{max} = \frac{wL^4}{76.81\, EI}$$

Exact solution of maximum deflection of beam,

$$y_{max} = \frac{5}{384}\frac{wL^4}{EI} = \frac{wL^4}{76.8\, EI}$$

(Almost equal to the exact solution).

Determination of bending moment at mid-span.

Bending moment, $m = EI \frac{d^2y}{dx^2}$.

WKT, $\frac{d^2y}{dx^2} = -\frac{a_1\pi^2}{l^2}\sin\frac{\pi x}{l} - \frac{9a_2\pi^2}{l^2}\sin\frac{3\pi x}{l}$

$$\therefore M = EI\left[-\frac{a_1\pi^2}{l^2}\sin\frac{\pi x}{l} - \frac{9a_2\pi^2}{l^2}\sin\frac{3\pi x}{l}\right]$$

$$= \frac{-EI\pi^2}{l^2}\left[a_1\sin\frac{\pi x}{l} + 9a_2\sin\frac{3\pi x}{l}\right]$$

Substituting a_1 & a_2 value.

$$M = -\frac{EI\pi^2}{l^2}\left[\frac{4WL^4}{EI\pi 5}\right]\left(\sin\frac{\pi x}{l} + \frac{9}{243}\sin\frac{3\pi x}{l}\right)$$

maximum bending moment at $x = l/2$.

$$M_{max} = \frac{-4WL^2}{\pi 3}\left[\sin\left(\frac{\pi}{l}\times\frac{l}{2}\right) + \frac{9}{243}\left(\sin\frac{3\pi}{2}\times\frac{l}{2}\right)\right]$$

$$= -\frac{4WL^2}{\pi 3}\left[1 - \frac{9}{243}\right]$$

$\sin\frac{\pi}{2} = 1$

$\sin\frac{3\pi}{2} = -1$

$$= \frac{-3.852\,WL^2}{\pi 3} = \frac{-WL^2}{\left(\frac{\pi 3}{3.852}\right)}$$

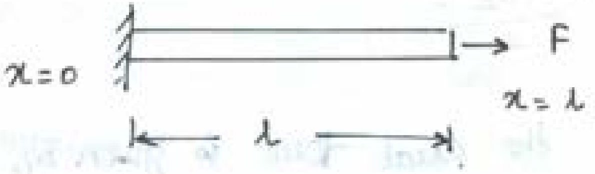
$$= \frac{-WL^2}{8.05}$$

$M_{max} = WL^2/8$. (Exact solution).

(Almost equal to the exact solution).

A bar of uniform cross-section is fixed at one end and left free at the other end and it is subjected to a uniform axial load F as shown in fig.

Calculate the displacement and stress using Rayleigh-Ritz procedure with two term polynomial function. Also compare the solution with the exact values.



Solution:-

displacement interms of polynomial function.

$$u = a_1 + a_2 x + a_3 x^2 + \cdots$$

Boundary condition, $x=0 + u=0$.

For two term polynomial function, $u = a_1 + a_2 x$

Applying boundary condition.

at $x=0$; $u=0$; $a_1 = 0$

Hence, $u = a_2 x$

As per rayleigh Ritz method,

The selected function will become acceptable approximate solution when it minimizes the total potential energy,

$$\pi = U - W$$

Strain energy, $U = \frac{EA}{2} \int_0^l \left(\frac{du}{dx}\right)^2 . dx$

for bar.

Now, $u = a_2 x$.

$$\frac{du}{dx} = a_2 \; ; \qquad \left(\frac{du}{dx}\right)^2 = a_2^2$$

Hence, $U = \frac{EA}{2} \int_0^l a_2^2 \, dx$

$$= \frac{EA a_2^2}{2} (l) = \frac{EA a_2^2 l}{2}$$

$$U = \frac{EA a_2^2 L}{2}$$

workdone by the axial force is given by,

$$W = \int_0^l F \cdot dx = \int_0^l \rho u A \cdot dx$$

$$= \rho A \int_0^l a_2 x \cdot dx$$

$$= \rho A a_2 \left[\frac{x^2}{2}\right]_0^l = \frac{\rho A a_2 l^2}{2}$$

$$W = \frac{\rho A a_2 L^2}{2}$$

Substituting in the potential energy equation,

$$\pi = \frac{EA a_2^2 L}{2} - \frac{\rho A a_2 L^2}{2}$$

minimum potential energy conditions, $\frac{d\pi}{da_2} = 0$.

$$\frac{d\pi}{da_2} = \frac{2EA a_2 L}{2} - \frac{\rho A L^2}{2} = 0.$$

$$a_2 = \frac{\rho L}{2E}$$

$\therefore$ deflection, $u = a_2 x = \frac{\rho L}{2E} \cdot x$

At $x = 0$; $u = 0$

$x = l$; $u = \frac{\rho L}{2E} \cdot l = \frac{\rho L^2}{2E}$.

Elongation of the bar, $du = u(l) - u(0)$

$$= \frac{\rho L^2}{2E}$$

Strain, $e = \frac{du}{dx} = \frac{\rho L^2/2E}{l} = \frac{\rho L}{2E}$

Stress, $\sigma = E \cdot e = E \cdot \frac{\rho L}{2E} = \frac{\rho L}{2}$

Actual extension:

$$\delta L = \int_0^l \frac{F \cdot dx}{AE} = \int_0^l \frac{\rho A x}{AE} \cdot dx$$

$$= \frac{\rho}{E} \int_0^l x \cdot dx = \frac{\rho}{E} \left[\frac{x^2}{2}\right]_0^l$$

$$\therefore \delta L = \frac{\rho L^2}{2E}$$

Approximate solution is equal with the exact solution.

—x—

Ritz method based on Integral Approach:-

Physical problem in terms of diff. Equation as, $D \frac{d^2y}{dx^2} + Q = 0$;

with boundary conditions $y(0) = y_0$ and $y(l) = y_l$

The above differential equation can be written as,

$$I = \int_0^l \left[\frac{D}{2} \left(\frac{dy}{dx}\right)^2 - Qy\right] dx$$

$I \to$ Termed as the functional; $\frac{\partial I}{\partial a} = 0$;

Problems based on Integral Approach:

Using variational Method, solve the given differential equation,

$$EI \frac{d^2y}{dx^2} - M(x) = 0$$ with the

boundary conditions $y(0) = 0$ and $y(l) = 0$

Solution:

The given differential equation is,

$$EI \frac{d^2y}{dx^2} - M(x) = 0.$$

with the boundary conditions $y(0) = 0$; $y(l) = 0$;

Expression interms of an equivalent integral form.

Functional, $I = \int_0^l \left[\frac{EI}{2}\left(\frac{dy}{dx}\right)^2 + M(x)y\right] dx$

The deflection function as,

$$y = a \sin \frac{\pi x}{l}$$

$$\frac{dy}{dx} = a\left(\cos \frac{\pi x}{l}\right)\left(\frac{\pi}{l}\right)$$

$$= \frac{a\pi}{l} \cos\left(\frac{\pi x}{l}\right)$$

$$\left(\frac{dy}{dx}\right)^2 = \frac{a^2\pi^2}{l^2} \cos^2 \frac{\pi x}{l}$$

$\Big| \cos^2\theta = \frac{1}{2} \times (1+\cos 2\theta)$

$$= \frac{a^2\pi^2}{l^2}\left[\frac{1}{2}\left(1+\cos \frac{2\pi x}{l}\right)\right]$$

functional, $$I = \int_0^l \frac{EI}{2}\left[\frac{a^2\pi^2}{l^2}\left\{\frac{1}{2}\left(1+\cos\frac{2\pi x}{l}\right)\right\}\right] + ma\sin\frac{\pi x}{l}\Bigg]\,dx.$$

$$= \int_0^l \frac{EIa^2\pi^2}{4l^2}\left(1+\cos\frac{2\pi x}{l}\right)dx + ma\int_0^l \sin\frac{\pi x}{l}\,dx$$

$$= \frac{EIa^2\pi^2}{4l^2}\left[x + \frac{l}{2\pi}\sin\frac{2\pi x}{l}\right]_0^l + ma\left[-\frac{l}{\pi}\cos\frac{\pi x}{l}\right]_0^l$$

$$= \frac{EIa^2\pi^2}{4l^2}\left[(l-0) + \frac{l}{2\pi}\left[\sin(2\pi) - \sin 0\right]\right] - \frac{mal}{\pi}(\cos\pi - \cos 0)$$

$$= \frac{EIa^2\pi^2 l}{4l^2} - \frac{mal}{\pi}(-1-1)$$

$$I = \frac{EIa^2\pi^2}{4l} + \frac{2mal}{\pi}$$

$$\frac{\partial I}{\partial a} = 0; \qquad \frac{\partial I}{\partial a} = \frac{2EIa\pi^2}{4l} + \frac{2\pi l}{\pi} = 0$$

$$a = -\frac{2ml}{\pi} \times \frac{4l}{2EI\pi^2} = -\frac{4ml^2}{\pi^3 EI}$$

Hence, the deflection function,

$$y = a\sin\frac{\pi x}{l} = -\frac{4ml^2}{\pi^3 EI}\sin\frac{\pi x}{l}$$

$$y = \frac{4mL^2}{\pi^3 EI}\sin\frac{\pi x}{l}$$

(-ve sign indicates the deflection in downwa consider the differential equation $\frac{d^2y}{dx^2} + 400x^2 = 0$; for $0 \le x \le 1$ subjected to boundary conditions $y(0) = 0$; $y(1) = 0$. The functional corresponding to the problem, to be extremized is given by,

$$I = \int_0^1 \left\{ -0.5 \left(\frac{dy}{dx}\right)^2 + 400\, x^2 y \right\} dx.$$

Find the solution of the problem using Rayleigh-Ritz method by considering a two-term solution as,

$$y(x) = a_1 x(1-x) + a_2 x(1-x^3).$$

Solution,

$$\frac{d^2y}{dx^2} + 400x^2 = 0 \quad \text{for} \quad 0 \le x \le 1.$$

Boundary conditions are $y = 0$ for $x = 0$ & $x = 1$;

Functional, $I = \int_0^l \left\{ -0.5 \left(\frac{dy}{dx}\right)^2 + 400\, x^2 y \right\} dx$

Trial solution,

$$y = a_1 x(1-x) + a_2 x(1-x^3)$$

$$= a_1 (x - x^2) + a_2 (x - x^4)$$

The trial solution satisfies the boundary conditions as $y = 0$ for $x = 0$ and $x = 1$.

$$\frac{dy}{dx} = a_1 (1 - 2x) + a_2 (1 - 4x^3)$$

$$\left(\frac{dy}{dx}\right)^2 = \left[a_1\ (1-2x) + a_2\ (1-4x^3)\right]^2$$

$$= \left[a_1 (1-2x)\right]^2 + \left[a_2 (1-4x^3)\right]^2 +$$

$$\left[2a_1\ (1-2x)\ a_2\ (1-4x^3)\right]$$

$$= a_1^2\ (1+4x^2-4x) + a_2^2\ (1+16x^6-8x^3) +$$

$$2a_1a_2\ (1-2x-4x^3+8x^4)$$

Substituting the values in the functional, I

$$I = \int_0^1 -0.5\left\{ a_1^2\ (1+4x^2-4x) + a_2^2\ (1+16x^6-8x^3) + \right.$$

$$\left. 2a_1a_2\ (1-2x-4x^3+8x^4)\right\} +$$

$$400x^2\ (\ a_1\ (x-x^2) + a_2\ (x-x^4)\}\ dx$$

$$= -0.5 \left\{ \quad a_1^2 \left[x + \frac{4x^3}{3} - \frac{4x^2}{2}\right]_0^1 + \right.$$

$$a_2^2 \left[x + \frac{16x^7}{7} - \frac{8x^4}{4}\right]_0^1 +$$

$$2a_1a_2 \left[x - \frac{2x^2}{2} - \frac{4x^4}{4} + \frac{8x^5}{5}\right]_0^1 +$$

$$400 \left[\ a_1 \left[\frac{x^4}{4} - \frac{x^5}{5}\right]_0^1 + a_2 \left[\frac{x^4}{4} - \frac{x^7}{7}\right]_0^1\ \right]$$

$$= \frac{-a_1^2}{6} - \frac{9a_2^2}{14} - \frac{3a_1a_2}{5} + 20a_1 + \frac{300a_2}{7}$$

For extremum conditions, $\frac{\partial I}{\partial a_1} = 0;\quad \frac{\partial I}{\partial a_2} = 0;$

$$\frac{\partial I}{\partial a_1} = -\frac{1}{3}a_1 - \frac{3}{5}a_2 + 20 = 0$$

$$5a_1 + 9a_2 = 300 \quad \text{—} ①$$

$$\frac{\partial I}{\partial a_2} = -\frac{9}{7}a_2 - \frac{3}{5}a_1 + \frac{300}{7} = 0$$

$$7a_1 + 15a_2 = 500 \quad \text{—} ②$$

solving, we get,

$$a_1 = 0; \quad a_2 = \frac{100}{3};$$

Hence the Solution,

$$y = 0 + \frac{100}{3}x(1-x^3)$$

$$y = \frac{100}{3}x(1-x^3)$$

Checking:-

$$y = \frac{100}{3}x(1-x^3)$$

$$\frac{dy}{dx} = \frac{100}{3}(1-4x^3)$$

$$\frac{d^2y}{dx^2} = \frac{100}{3}(0-12x^2) = -400x^2$$

Given differential equation

$$\frac{d^2y}{dx^2} + 400x^2 = -400x^2 + 400x^2 = 0.$$

Hence, the solution is the exact solution.

In weighted residual method, the error is minimized by the four methods.

- * Point collocation method
- * Sub-domain collocation method
- * Least square method
- * Galerkin's method.

Consider $y(x)$ is the exact solution for the differential equation.

Then, approximate function (trial function) must be considered.

$$y(x) = f(x, a_i) \; ; \; i = 1, 2 \ldots$$

and, the residual $R(x, a_i)$ will be found out.

✓ The trial function must satisfy the boundary conditions.

The number of weighting functions is equal to the number of unknown coefficients in the approximate function.

a) Point collocation method:

The residual $R(x, a_i)$ is set equal to zero.

weighted function, $W_i = \delta(x - x_i)$ is expressed,

$$\therefore \int_D W_i R(x, a_i)\, dx = 0$$

$$\int_D \delta(x - x_i)\; R(x, a_i).dx = 0;$$

At point, $x = x_i$; $W_i = 1$ hence $R(x, a_i) = 0$;

At other points in the domain, $W_i = 0$.

b) sub-domain collocation method:

domain is subdivided into n sub domain and the integral of the residual over each sub domain is required to be zero.

$$\int_D R(x, a_i).\, dx = 0 \qquad W_i = 1 \text{ (over the domain)}.$$

c) Least square method:

The integral of the weighted square of the residual over the domain is required to be minimum

$$I = \int_D \left[R(x, a_i) \right]^2 dx = \text{minimum}.$$

and $\dfrac{\partial I}{\partial a_i} = 0$; $i = 1, 2 \dots n$

d) Galerkin's method:

Trial function $y(x)$ itself is considered as weight function.

$$\int_D W_i \ R(x, a_i) \cdot dx = \int_D y(x) \ R(x, a_i) \cdot dx = 0$$

---

consider the differential equations for a Problem as,

$$\frac{d^2y}{dx^2} + 300 x^2 = 0 ; \qquad 0 \leq x \leq 1$$

with the boundary conditions $y(0) = 0$; $y(1) = 0$.

find the solution of the Problem using a one co-efficient trial function as $y = a_1 x (1 - x^3)$. use.

i) Pt collocation method ii) Sub-domain collocation method

iii) Least square method iv) Galerkin's method.

Solution:

Given, differential equation,

$$\frac{d^2y}{dx^2} + 300 x^2 = 0 ; \qquad 0 \leq x \leq 1 \quad \text{— (1)}$$

The boundary conditions are $y = 0$ at $x = 0$ & $x = 1$.

The trial function, $y = a_1 x (1 - x^3)$ — (2)

Checking of trial function whether it will satisfy the boundary conditions or not.

for, $x = 0$; $y = a_1 (0) (1 - 0) = 0$

$x = 1$; $y = a_1 (1) (1 - 1^3) = 0$;

Hence, the selected trial function satisfies the boundary conditions.

Equ (2) ⇒ $y = a_1 \ (x - x^4)$ $\quad | x^n \rightarrow n x^{n-1}$

$$\frac{dy}{dx} = a_1 \ (1 - 4x^3)$$

$$\frac{d^2y}{dx^2} = a_1 \ (0 - 12x^2) = -12 a_1 x^2.$$

Substituting in Equ(1) we get,

Residual, $R = -12 a_1 x^2 + 300 x^2.$

i) Point collocation method.

$$R = -12 a_1 x^2 + 300 x^2 = 0.$$

One collocation point is required in between 0 to 1.

let, $x = 1/2$ (in between)

Then, $R = -12 a_1 (1/2)^2 + 300 (1/2)^2 = 0.$

$3a_1 = 75$ (or) $a_1 = 25$

Equ(2) ⇒ Trial function,

$$y = 25 x(1 - x^3)$$

ii) Sub-domain collocation method

$$\int_0^1 R dx = 0;$$

$$\int_0^1 (-12 a_1 x^2 + 300 x^2) \ dx = 0.$$ $\quad \int x^n = \frac{x^{n+1}}{n+1}$

$$\left[-12 a_1 \frac{x^3}{3} + 300 \frac{x^3}{3}\right]_0^1 = 0$$

$$-4a_1 + 100 = 0$$

$$a_1 = \frac{100}{4} = 25$$

Hence, trial function, $y = 25x(1-x^3)$

iii) Least Square Method:

The functional, $I = \int_0^1 R^2 \, dx = \text{minimum}$

$$I = \int_0^1 R^2 dx = \int_0^1 (-12a_1x^2 + 300x^2)^2 \, dx$$

$$= \int_0^1 (144\, a_1^2 x^4 + 90000\, x^4 - 7200\, a_1 x^4) \, dx$$

$$= \left[144\, a_1^2 \frac{x^5}{5} + 90000 \frac{x^5}{5} - 7200\, a_1 \frac{x^5}{5}\right]_0^1$$

$$I = \frac{144}{5} a_1^2 + \frac{90000}{5} - \frac{7200\, a_1}{5}$$

Now, $\frac{\partial I}{\partial a_1} = 0$;

$$\frac{\partial I}{\partial a_1} = \frac{288}{5} a_1 - \frac{7200}{5} = 0;$$

$$a_1 = 25$$

Hence the trial function, $y = 25x(1-x^3)$

iv) Galerkin's Method:

$$\int_0^1 (y . R) \, dx = 0$$

$$\int_0^1 a_1 x (1-x^3) (- 12 a_1 x^2 + 300 x^2) \, dx = 0;$$

$$\int_0^1 (a_1 x - a_1 x^4) (-12 a_1 x^2 + 300 x^2) \, dx = 0$$

$$\int_0^1 (-12 a_1^2 x^3 + 300 a_1 x^3 + 12 a_1^2 x^6 - 300 a_1 x^6) \, dx = 0.$$

$$\left[-12 a_1^2 \frac{x^4}{4} + 300 a_1 \frac{x^4}{4} + 12 a_1^2 \frac{x^7}{7} - 300 a_1 \frac{x^7}{7} \right]_0^1 = 0$$

$$- \frac{12 a_1^2}{4} + \frac{300 a_1}{4} + \frac{12}{7} a_1^2 - \frac{300}{7} a_1 = 0;$$

÷ by $(-12 a_1)$

$$\frac{a_1}{4} - \frac{25}{4} - \frac{a_1}{7} + \frac{25}{7} = 0$$

$$3a_1 = 75 \quad ; \qquad a_1 = 25$$

Trial function, $y = 25 x (1-x^3)$.

For the differential equations,

$$\frac{d^2y}{dx^2} + 400 x^2 = 0 ; \qquad 0 \leq x \leq 1$$

with the boundary conditions $y(0) = 0$ and $y(1) = 0$.

find the solution of the problem using a two term trial function by using

i) Point collocation Method

ii) Sub-domain collocation Method.

Solution.

Given differential Equation,

$$\frac{d^2y}{dx^2} + 400x^2 = 0; \quad 0 \le x \le 1 \text{ --- (1)}$$

Boundary conditions are, $y = 0$ at $x = 0$ & $x = 1$

let,
Select a trial function as,

$$y = a_1 x(1-x) + a_2 x(1-x^3) \text{ --- (2)}$$

$$y = a_1(x-x^2) + a_2(x-x^4)$$

differentiating,

$$\frac{dy}{dx} = a_1(1-2x) + a_2(1-4x^3)$$

$$\frac{d^2y}{dx^2} = a_1(-2) + a_2(-12x^2) = -2a_1 - 12a_2x^2$$

Substituting in Eq (1),

Residual, $R = -2a_1 - 12a_2x^2 + 400x^2$

Since,
there are two parameters (a_1 & a_2), the domain ranging from 0 to 1 can be divided into two subdomains, 0 to $\frac{1}{2}$ & $\frac{1}{2}$ to 1.

i) Point collocation Method:

In this method, the residual is set to zero.

$$R = -2a_1 - 12a_2x^2 + 400x^2 = 0; \text{ --- (3)}$$

Since, the number of co-efficient to be determined is two.

let, there two points be, $x = 1/4$ (for 0 to 1/2) +

$x = 3/4$ (for 1/2 to 1).

For $x = 1/4$;

Eqn(3) → $-2a_1 - 12 a_2 (1/4)^2 + 400 (1/4)^2 = 0$

$$-2a_1 - \frac{3}{4} a_2 + 25 = 0$$

$$8a_1 + 3a_2 = 100$$

For $x = 3/4$;

Eqn(3) → $R = -2a_1 - 12a_2 (3/4)^2 + 400 (3/4)^2 = 0$

$$-2a_1 - \frac{27}{4} a_2 + 225 = 0.$$

$$8a_1 + 27a_2 = 900$$

solving, the above two Equations, we get

$$a_1 = 0; \quad a_2 = 100/3 ;$$

Hence the trial function,

$$y = \frac{100}{3} x (1-x^3)$$

i) Subdomain collocation method.

For the Sub domain ($x = 0$ to $1/2$)

$$\int_0^{1/2} R . dx = 0 \rightarrow \int_0^{1/2} (-2a_1 - 12 a_2 x^2 + 400 x^2) dx = 0;$$

$$= \left[-2a_1 x - 12 a_2 \frac{x^3}{3} + 400 \frac{x^3}{3} \right]_0^{1/2}$$

$$= -a_1 - \frac{a_2}{2} + \frac{50}{3} = 0 \quad \text{(or)}$$

$$6a_1 + 3a_2 = 100$$

For the subdomain ($x = 1/2$ to 1)

$$\int_{1/2}^{1} R\,dx = 0; \quad \int_{1/2}^{1} (-2a_1 - 12a_2 x^2 + 100x^2)\,dx = 0$$

$$= \left[-2a_1 x - 12a_2 \frac{x^3}{3} + 100\frac{x^3}{3}\right]_{1/2}^{1}$$

$$= -a_1 - \frac{7}{2} a_2 + \frac{350}{3} = 0$$

$$6a_1 + 21a_2 = 700$$

Solving, we get

$$a_1 = 0; \quad a_2 = 100/3$$

Hence, Trial function, $y = \frac{100}{3} x(1-x^3)$

—.

(T-1) The differential equations of a physical phenomenon is given by $\frac{d^2y}{dx^2} - 10x^2 = 5$. Obtain the two term Galerkin solution by using the trial functions.

$N_1(x) = x(x-1)$; $N_2(x) = x^2(x-1)$; $0 \le x \le 1$

with Boundary conditions are $y(0) = 0$; $y(1) = 0$.

Solution.

Trial functions becomes eligible only if it satisfies the boundary conditions.

At $x = 0$; $y(x) = a_1 x(x-1) + a_2 x^2(x-1)$

& At $x = 1$ 　At $x = 0$; $y(x) = 0$

$x = 1$ 　$y(x) = 0$.

$y(x) = a_1 x(x-1) + a_2 x^2 (x-1).$

$\frac{dy}{dx} = a_1 (2x-1) + a_2 (3x^2 - 2x)$

$\frac{d^2y}{dx^2} = a_1 (2-0) + a_2 (6x^2 - 2)$

$= 2a_1 + 6x^2 a_2 - 2a_2$

Substituting the differential equation, $\frac{d^2y}{dx^2} - 10x^2 = 5$

Residual, $R = 2a_1 + 6x^2 a_2 - 2a_2 - 10x^2 = 5$

$= 2a_1 + 6x^2 a_2 - 2a_2 - 10x^2 - 5$

using Galerkin's method:

$$\int_0^1 w_i \; R \; dx = 0$$

Here, the weighting functions are the trial functions which are $x(x-1)$ and $x^2(x-1)$.

Hence, $\int_0^1 x(x-1) \; R \, . dx = 0 \quad ; \quad \int_0^1 x^2 (x-1) \; R \, . dx = 0$

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad$ L ① $\quad\quad\quad\quad\quad\quad\quad\quad$ L ②

solving Equ (1)

$$\int_0^1 (x^2 - x)(2a_1 + 6x^2 a_2 - 2a_2 - 10x^2 - 5) \, . dx = 0$$

$$\int_0^1 (2a_1 x^2 + 6x^4 a_2 - 2a_2 x^2 - 10x^4 - 5x^2)$$

$$- (2a_1 x) - 6x^3 a_2 + 2a_2 x + 10x^3 - 5x) \, . dx = 0$$

$$\int_0^1 (6a_2 - 10)x^4 + (-6a_2 + 10)x^3 + (2a_1 - 2a_2 - 5)x^2$$

$$+ (-2a_1 + 2a_2 - 5)x \; . \; dx = 0 .$$

$$(6a_2-10)\left[\frac{x^5}{5}\right]_0^1 + (-6a_2+10)\left[\frac{x^4}{4}\right]_0^1 +$$

$$(2a_1-2a_2-5)\left[\frac{x^3}{3}\right]_0^1 + (-2a_1+2a_2-5)\left[\frac{x^2}{2}\right]_0^1 = 0$$

$$\frac{(6a_2-10)}{5} + \frac{-6a_2+10}{4} + \frac{2a_1-2a_2-5}{3} + \frac{-2a_1+2a_2-5}{2} = 0$$

$$\frac{2a_1}{3} - \frac{2a_1}{2} + \frac{6a_2}{5} - \frac{6a_2}{4} - \frac{2a_2}{3} + \frac{2a_2}{2} - 2 + 2.5 - \frac{5}{3} - \frac{5}{2}$$

Soling,

$$a_1 + 0.5a_2 = 4 \text{ --- } (3)$$

Solving eqn (2),

$$\int_0^1 (x^3 - x^2)\,(2a_1 + 6a_2x - 2a_2 - 10x^2 - 5)\,.\,dx = 0;$$

$$\int_0^1 (2a_1x^3 + 6a_2x^4 - 2a_2x^3 - 10x^5 - 5x^3) +$$

$$(-2a_1x^2 - 6a_2x^3 + 2a_2x^2 + 10x^4 + 5x^2)\,dx$$

Solving,

$$a_1 + 0.8012\,a_2 = 4.518 \text{ --- } (4)$$

Solving eqn (3) & (4) we get,

$$a_1 = 3.140$$
$$a_2 = 1.719.$$

Hence, the two term approximate solution is,

$$y = 3.140x\,(x-1) + 1.719\,x^2(x-1)$$

$$y = 1.719x^3 + 1.421\,x^2 - 3.140x.$$

UNIT- II

ONE DIMENSIONAL PROBLEMS

1D second order Equations – Discretization – Element types – Linear + Higher Order Elements – Derivation of shape function + stiffness matrices + force vectors – Assembly of Matrices – Solution of Problems from solid mechanics + Heat transfer. Longitudinal vibration frequencies + mode shapes; fourth order Beam Equations – Transverse deflections + Natural freq. of Beams.

Classification of finite Element Problems :-

a)	Structural	Non structural.
b)	Linear ✓ Parameter varies in direct proportion	Non-linear (or) Higher Order Element. ✓ Parameter varies in Square proportion (or) cubic
c)	One dimensional ;	Two dimensional ; Three dimensional.
d)	simplex Problem	complex Problem.
e)	static Problem	Dynamic Problem.

Types of Solid Products :

1D – long length + width + Height is Negligible

Ex : Bars , Beams , Truss , shafts etc.

2D – Two dimensions are large + Third dimension is Negligible

Ex : Plates , sheet Metals .

3D – Three dimensions are large.

Ex : cylinder, cone, sphere.

Variable Problems:

✓ scalar - Quantity of variable have mangitude only.

✓ vector - Quantity of variable have both magnitude & direction.

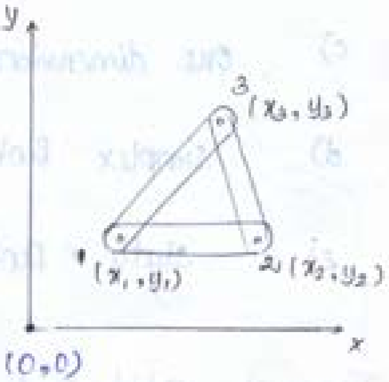
co-ordinate systems:

✓ location of various nodes & elements must be expressed with respect to some fixed axis for easy identification.

2 types of co-ordinate system:

i) Global co-ordinate system.

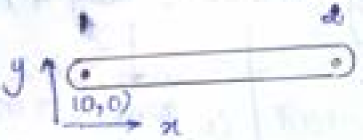
✓ various Nodes & Elements are specified by the common axes whose origin is mostly away from the Element.



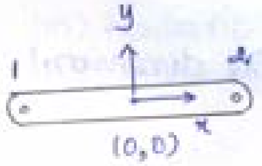
ii) Local co-ordinate system.

✓ co-ordinate axes are placed on the element

✓ If the origin is lying at the centre of the element & it's magnitude never exceed unity, then it's Natural co-ordinate system.



Local co-ordinate system.



Natural co-ordinate system

One - Dimensional Problems :-

Bar & Beam Elements are considered as one dimensional Elements. These Elements are often used to model trusses and frame structures.

- ✓ Bar is a member which resist only axial loads.
- ✓ Beam " " which resist axial, lateral + twisting loads.
- ✓ Truss is a assemblage of bars with pin joints.
- ✓ frame " " beam Elements.

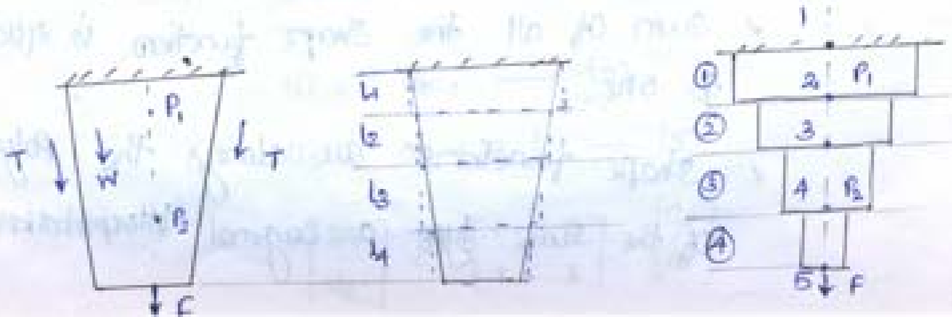
Finite Element Modeling:

✓ Any analysis will be easy if the analysing Part has uniform (or) simple structure.

✓ In Practice, all the systems are not having uniform structure.

✓ System must be modeled to have uniform Properties.

Ex: Taper Rod



a) One dimensional bar - loaded by Traction, body & pt loads

Force acts on volume of body Ex: Self wt

b) Method of modelling

Force acting on the surface Ex: Friction, Shear

c) Finite Element Model

4 Element - ①, ②, ③ & ④

5 Nodes - 1, 2, 3, 4, 5

In 1-D Problems, every node is permitted to displace in one direction only.

✓ Each node will have one displacement, so. this finite Element model has 5 D.O.F (5 displacements)

Shape function:

Shape function is a mathematical expression used for finding the values of variable inside the element by means nodal values of the same variable.

✓ They are used to express the geometry (or) shape of the element.

Properties of shape functions are,

✓ Each shape function has a value of one at its own node & zero at the other node.

✓ Sum of all the shape function is equal to one.

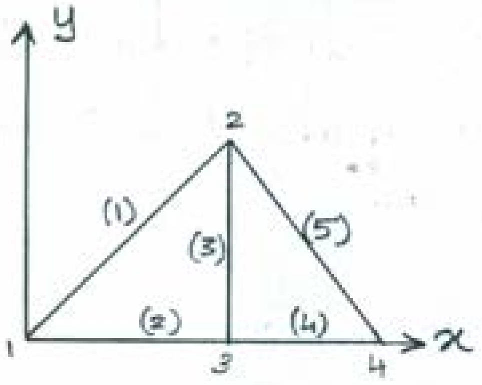
✓ Shape functions are always the polynomial of the same type as original interpolation equation.

Coordinate Systems

Coordinates are used to define the location of points in the element.

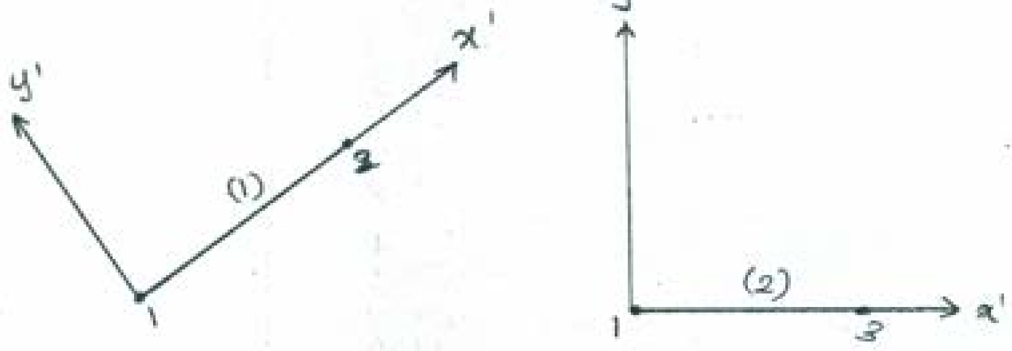
Global Coordinates

The coordinate system used to define the points in the entire structure is called global coordinate system.



Local Coordinates

In the initial stages of finite element formulation for the convenience of deriving element properties, for each element a seperate coordinate system is used. However the final equations are to be formed in the common coordinate system ie global coordinate systems only.



Natural Coordinates

A natural coordinate system is a coordinate system which permits the specification of a point within the element by a set of dimensionless numbers, whose magnitude never exceeds unity. It is obtained by assigning weightages to the nodal coordinates in defining the coordinate of any point inside the element. Hence such system has the property that i^{th} coordinate has unit value at node i of the element and zero value at all other nodes.

Natural coordinates in x

Natural coordinates in ξ

Stiffness matrix

Let $u_1, u_2, u_3, \ldots u_n$ be the nodal displacements for the nodal loads $F_1, F_2, F_3 \ldots F_n$

$$\{u\} = \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_n \end{Bmatrix} \; ; \{F\} = \begin{Bmatrix} F_1 \\ F_2 \\ \vdots \\ F_n \end{Bmatrix}$$

We know

$$\{F\} = [K]\{u\}$$

We know

External work done P = Internal strain Energy U

Considering external work done P

$$P = \tfrac{1}{2} F_1 u_1 + \tfrac{1}{2} F_2 u_2 + \tfrac{1}{2} F_3 u_3 + \ldots\ldots + \tfrac{1}{2} F_n u_n$$

$$P = \tfrac{1}{2} [F_1, F_2 \ldots\ldots F_n] \begin{Bmatrix} u_1 \\ \vdots \\ u_n \end{Bmatrix}$$

$$P = \tfrac{1}{2} \{F\}^T \{u\}$$

$$P = \tfrac{1}{2} \{[K]\{u\}\}^T \{u\}$$

$$P = \tfrac{1}{2} [K]^T \{u\}^T \{u\}$$

$[K]^T = [K]$ $\quad$ ∵ $[K]$ is a square & symmetric matrix

∴ External Work done $P = \tfrac{1}{2} [K] \{u\}^T \{u\}$ —— ①

Considering internal strain energy U

$$U = \int_V \tfrac{1}{2} \{\epsilon\}^T \{\sigma\} \, dv$$

From the strain displacement relation, we know

$$\{e\} = [B]\{u\}$$

$$\{\epsilon\}^T = [B]^T \{u\}^T$$

From the stress strain relation, we know

$$\{\sigma\} = [D]\{\epsilon\}$$

Substituting in internal strain energy U

$$U = \int_V \tfrac{1}{2} [B]^T \{u\}^T [D] \{\epsilon\} \, dv$$

$$U = \tfrac{1}{2} \{u\}^T \int_V [B]^T [D] \{\epsilon\} \, dv$$

$$U = \tfrac{1}{2} \{u\}^T \int_V [B]^T [D] \{[B]\{u\}\} \, dv$$

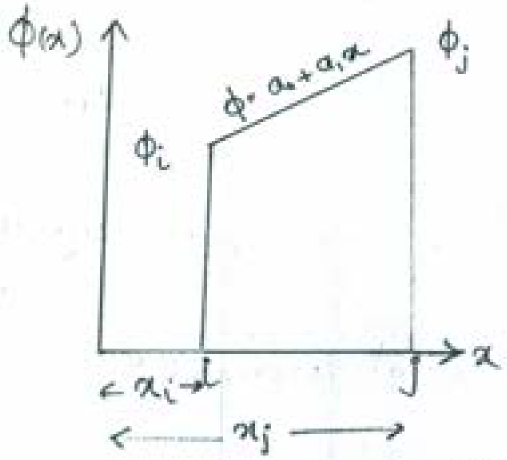
$$U = \tfrac{1}{2} \{u\}^T \{u\} \left[\int_V [B]^T [D] [B] \, dv \right] \text{ —— ②}$$

Comparing P & U, ① = ②

$$\cancel{\tfrac{1}{2}} [K] \cancel{\{u\}^T} \cancel{\{u\}} = \cancel{\tfrac{1}{2}} \cancel{\{u\}^T} \cancel{\{u\}} \left[\int_V [B]^T [D] [B] \, dv \right]$$

$$\boxed{[K] = \int_V [B]^T [D] [B] \, dv}$$

Shape function for One-Dimensional Linear Element in Global Coordinates



Let the approximating function be

$$\phi = a_0 + a_1 x$$

The nodal conditions are given by @ $x = x_i$, $\phi = \phi_i$
@ $x = x_j$, $\phi = \phi_j$

Substituting the nodal conditions in $\phi = a_0 + a_1 x$

$$\phi_i = a_0 + a_1 x_i$$
$$\phi_j = a_0 + a_1 x_j$$

Solving the above eq. for a_0 & a_i

$$a_0 = \frac{\phi_i x_j - \phi_j x_i}{x_j - x_i} \; ; \; a_1 = \frac{\phi_j - \phi_i}{x_j - x_i}.$$

Substituting a_0 & a_1 values in $\phi = a_0 + a_1 x$

$$\phi = \frac{x_j \phi_i - \phi_j x_i}{x_j - x_i} + \left(\frac{\phi_j - \phi_i}{x_j - x_i}\right) x$$

$$= \frac{\phi_i x_j - \phi_j x_i}{\ell} + \left(\frac{\phi_j - \phi_i}{\ell}\right) x$$

$$= \frac{\phi_i x_j - \phi_j x_i}{\ell} + \frac{\phi_j x - \phi_i x}{\ell}$$

$$= \left(\frac{x_j - x}{\ell}\right)\phi_i + \left(\frac{x - x_i}{\ell}\right)\phi_j$$

$$\phi = N_1 \phi_i + N_2 \phi_j$$

$$N_1 = x_j - x \; ; \; N_2 = \frac{x - x_i}{}$$

Shape function for One dimensional bar element in local coordinates

Let $u = a_0 + a_1 x = \langle 1 \;\; x \rangle \begin{Bmatrix} a_0 \\ a_1 \end{Bmatrix}$ — ①

The nodal conditions are given by @ $x=0$, $u=u_i$
@ $x = l$, $u = u_j$

Substituting the nodal conditions

at $x=0$ $\quad u_i = a_0 + a_1(0)$

$x = l$ $\quad u_j = a_0 + a_1(l)$

$$\begin{Bmatrix} u_i \\ u_j \end{Bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & l \end{bmatrix} \begin{Bmatrix} a_0 \\ a_1 \end{Bmatrix}$$

$$\begin{Bmatrix} a_0 \\ a_1 \end{Bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & l \end{bmatrix}^{-1} \begin{Bmatrix} u_i \\ u_j \end{Bmatrix} \quad \text{— ②}$$

Sub ② in ①

$$\{u\} = \langle 1 \;\; x \rangle \begin{bmatrix} 1 & 0 \\ 1 & l \end{bmatrix}^{-1} \begin{Bmatrix} u_i \\ u_j \end{Bmatrix}$$

$$= \langle 1 \;\; x \rangle \begin{bmatrix} 1 & 0 \\ -1/l & 1/l \end{bmatrix} \begin{Bmatrix} u_i \\ u_j \end{Bmatrix}$$

$$= \left\langle 1 - \frac{x}{l} \;\; \frac{x}{l} \right\rangle \begin{Bmatrix} u_i \\ u_j \end{Bmatrix}$$

$$\{u\} = \langle N_1 \;\; N_2 \rangle \begin{Bmatrix} u_i \\ u_j \end{Bmatrix}$$

$$\{u\} = N_1 u_i + N_2 u_j$$

where $N_1 = 1 - \frac{x}{l}$

$N_2 = \frac{x}{l}$

Shape function for One Dimensional bar element in natural coordinates (ξ, η)

- Coordinate values between −1 & 1
- Convenient form for numerical integration (Gauss Quadrature Method)
- To device curve sided element (Isoparametric formulation)

For node 1

$$\frac{x}{a} + \frac{y}{b} = 1$$

$$\frac{\xi}{1} = 1$$

$$\xi - 1 = 0$$

$$N_1 = C_1(\xi - 1)$$

at node 1

$$N_1 = 1$$

$$\xi = -1$$

$$1 = C_1(-1-1)$$

$$1 = -2C_1$$

$$C_1 = -1/2$$

$$N_1 = -1/2(\xi - 1)$$

$$N_1 = \frac{1-\xi}{2}$$

For node 2

$$\frac{x}{a} + \frac{y}{b} = 1$$

$$\frac{\xi}{-1} = 1$$

$$-\xi = 1$$

$$\xi + 1 = 0$$

$$N_2 = C_2(\xi + 1)$$

at node 2

$$N_2 = 1$$

$$\xi = 1$$

$$N_2 = C_2(1+1)$$

$$1 = C_2(2)$$

$$C_2 = 1/2$$

$$N_2 = 1/2(\xi + 1)$$

$$N_2 = \frac{1+\xi}{2}$$

Properties of shape function

1) $(N_i)_j = 1$ if $i = j$

$= 0$ if $i \neq 0$

j = node no.

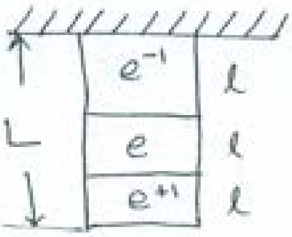
i = shape fn. no.

2) $\sum_{j=0}^{n} N_j = 1$

Stiffness Matrix for a structural problem with one dimensional element using Ritz Equation

Ritz Equation

$$\int_0^l EA(x)\,\frac{du^*}{dx}\cdot\frac{d\psi}{dx}\,dx = \int_0^l \gamma A(x)\,\psi^*(x)\,dx + P\psi(L)$$



Considering element 'e'

$$\int_0^l E^e A^e(x)\,\frac{du^*}{dx}\cdot\frac{d\psi}{dx}\,dx = \int_0^l \gamma^{(e)} A^e(x)\,\psi^e(x)\,dx + P^{(e)}\psi(e)$$

Let the approximate solution be

$u^{*e} = a_0 + a_1 x$

We need to substitute the boundary conditions to make approximate solution $\bar{u}^*$, eligible approx. solution $\bar{u}^*$. But the boundary conditions at the element level is not known. Now by introducing nodes i & j.

We need to relate the displacement u with the nodal displacement u_i & u_j

$$u = \begin{Bmatrix} u_i \\ u_j \end{Bmatrix} = \langle 1 \quad x \rangle \begin{Bmatrix} a_0 \\ a_1 \end{Bmatrix}$$

@ $x = 0$ $\quad u_i = a_0 + a_1 0$

$x = l$ $\quad u_j = a_0 + a_1 l$

$$\begin{Bmatrix} u_i \\ u_j \end{Bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & l \end{bmatrix} \begin{Bmatrix} a_0 \\ a_1 \end{Bmatrix}$$

$$\begin{Bmatrix} a_0 \\ a_1 \end{Bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & l \end{bmatrix}^{-1} \begin{Bmatrix} u_i \\ u_j \end{Bmatrix}$$

$$u = \langle 1 \quad x \rangle \begin{bmatrix} 1 & 0 \\ 1 & l \end{bmatrix}^{-1} \begin{Bmatrix} u_i \\ u_j \end{Bmatrix}$$

$$u = \langle 1 - \frac{x}{l} \quad \frac{x}{l} \rangle \begin{Bmatrix} u_i \\ u_j \end{Bmatrix}$$

$$u = \langle N_1 \quad N_2 \rangle \begin{Bmatrix} u_i \\ u_j \end{Bmatrix} \qquad N_1 = 1 - \frac{x}{l} \qquad N_2 = \frac{x}{l}$$

$$u = N_1 u_i + N_2 u_j$$

The field variable for element u^{*e} is related to the nodal field variable u_i & u_j through functions called as shape functions N_1 & N_2

$$u^{*e} = N_1 u_1 + N_2 u_2$$

$$= \sum_{i=1}^{2} N_i u_i \qquad i = 1, 2$$

$$\psi^e \sum_{j=1}^{2} N_j \qquad j = 1, 2$$

$$\leq \int_0^l E^e A^e \frac{dN_i}{dx} \cdot \frac{dN_j}{dx} \, dx$$

When $i = 1, j = 1$

$$K_{11}^e = \int_0^l E^e A^e \frac{d}{dx}\left(1 - \frac{x}{l}\right) \cdot \frac{d}{dx}\left(1 - \frac{x}{l}\right) dx$$

$$= E^e A^e \int_0^l \left(-\frac{1}{l}\right)\left(-\frac{1}{l}\right) dx$$

$$= \frac{A^e E^e}{l^2} \left[x\right]_0^l \quad = K_{11} = \frac{A^e E^e}{l}$$

When ~~N~~ i = 1 , j = 2

$$E^eA^e\int_0^l \frac{d}{dx}\left(1-\frac{x}{l}\right)\frac{d}{dx}\left(\frac{x}{l}\right)dx$$

$$E^eA^e\int\left(-\frac{1}{l}\right)\left(\frac{1}{l}\right)dx$$

$$-\frac{E^eA^e}{l^2}\left[x\right]_0^l$$

$$K_{12} = -\frac{A^eE^e}{l}$$

$$K_{12} = K_{21}$$

When ~~At~~ i = 2, j = 2

$$E^eA^e\int_0 \frac{dN_2}{dx}\frac{dN_2}{dx}\,dx$$

$$E^eA^e\int_0^l \frac{d}{dx}\left(\frac{x}{l}\right)\frac{d}{dx}\left(\frac{x}{l}\right)dx$$

$$E^eA^e\int_0^l \left(\frac{1}{l}\right)\left(\frac{1}{l}\right)dx$$

$$\frac{E^eA^e}{l^2}\left[x\right]_0^l$$

$$K_{22} = \frac{A^eE^e}{l}$$

$$[K]\begin{bmatrix}K_{11} & K_{12}\\ K_{21} & K_{22}\end{bmatrix} = \begin{bmatrix}\frac{A^eE^e}{l^e} & -\frac{A^eE^e}{l^e}\\ -\frac{A^eE^e}{l} & \frac{A^eE^e}{l^e}\end{bmatrix}$$

$$[K] = \frac{A^eE^e}{l^e}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}$$

Stiffness Matrix for an One Dimensional Bar Element

For bar element

$$u = N_1 u_1 + N_2 u_2$$

$$u = [N_1 \;\; N_2] \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} \quad ; \quad N_1 = 1 - \frac{x}{l}$$

$$N_2 = \frac{x}{l}$$

$$\varepsilon = \frac{du}{dx} = \frac{d}{dx} [N_1 \;\; N_2] \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$\epsilon = \begin{bmatrix} \frac{dN_1}{dx} & \frac{dN_2}{dx} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$[B] = \begin{bmatrix} \frac{dN_1}{dx} & \frac{dN_2}{dx} \end{bmatrix} = \begin{bmatrix} -1/l & 1/l \end{bmatrix}$$

$$\{\sigma\} = \sigma = [D]\{\epsilon\} \quad \text{for 1D} \quad \sigma = E\epsilon$$

$$\Rightarrow [D] = E$$

Stiffness matrix $[k] = \iiint_v [B]^T [D] [B] \, dv$

$$[k] = \int_0^l \begin{Bmatrix} -1/l \\ 1/l \end{Bmatrix} \times E \times \begin{bmatrix} -1/l & 1/l \end{bmatrix} dv$$

$$[k] = \int_0^l \begin{Bmatrix} -1/l \\ 1/l \end{Bmatrix} \times E \times \begin{bmatrix} 1/l & 1/l \end{bmatrix} A \, dx$$

$$= EA \int_0^l \begin{bmatrix} 1/l^2 & -1/l^2 \\ -1/l^2 & 1/l^2 \end{bmatrix} dx$$

$$= EA \begin{bmatrix} 1/l^2 & -1/l^2 \\ -1/l^2 & 1/l^2 \end{bmatrix} [x]_0^l$$

$$= EA\ell \times \frac{1}{l^2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

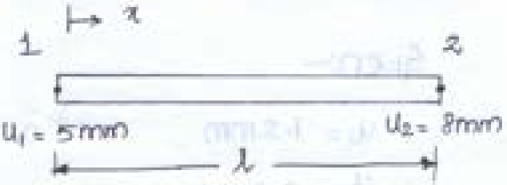
$$[k] = \frac{EA}{l} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Problems:-

A two noded bar element is shown in the fig below. The nodal displacements are $u_1 = 5$ mm and $u_2 = 8$ mm. Calculate the displacement at $x = l/4$, $l/3$ & $l/2$.

Given:-

$u_1 = 5$ mm; $u_2 = 8$ mm.



Solution:-

Displacement function, $u = N_1 u_1 + N_2 u_2$

$$N_1 = \frac{l - x}{l} \; ; \; N_2 = \frac{x}{l}$$

a) displacement @ $x = l/4$

$$u = \left(\frac{l - l/4}{l}\right) \times 5 + \left(\frac{l/4}{l}\right) \times 8$$

$$u = 5.75 \text{ mm.}$$

b) displacement @ $x = l/3$

$$u = \left(\frac{l - l/3}{l}\right) \times 5 + \left(\frac{l/3}{l}\right) \times 8$$

$$= 6 \text{ mm}$$

c) displacement @ $x = l/2$

$$u = \left(\frac{l - l/2}{l}\right) 5 + \left(\frac{l/2}{l}\right) 8$$

$$= 6.5 \text{ mm.}$$

Result:-

Displacement function

@ $x = l/4$; $u = 5.75$ mm

@ $x = l/3$: $u = 6$ mm

@ $x = l/2$; $u = 6.5$ mm

A rod of diameter 10 mm, length 200 mm has Nodal displacements due to axial loads as 1.2 mm and 2.8 mm. The position of rod is shown in the fig.

Calculate i) displacement at point Q, ii) strain iii) Stress. iv) strain Energy for the rod. Take E = 210GPa.

Given:-

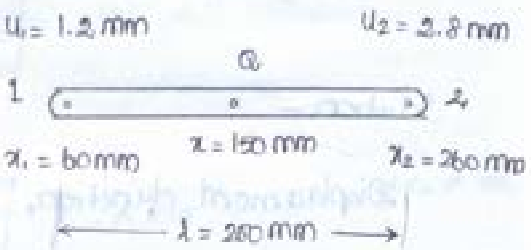
$u_1 = 1.2 mm$

$u_2 = 2.8 mm$

$l = 200 mm$; $d = 10 mm$

$E = 210 GPa = 210 \times 10^9 N/m^2$

$= 210 \times 10^3 N/mm^2$



Solution:-

i) displacement at point Q:

distance Q from Point 1

$x = 150 - 60 = 90$

$$u_{@Q} = N_1 u_1 + N_2 u_2$$

$$N_1 = \frac{l - x}{l}; \quad N_2 = \frac{x}{l}$$

$$N_1 = \frac{200 - \overset{90}{150}}{200}; \quad N_2 = \frac{\overset{90}{150}}{200}$$

$$= \overset{0.55}{0.25}; \quad = \overset{0.45}{0.75}$$

$$u_{@Q} = 0.2\overset{5}{5}(1.2) + 0.\overset{4}{7}5(2.8)$$

$$= \overset{1.92}{2.26} mm$$

ii) strain, $e = [B]\{u\}$

$$[B] = \left[\frac{-1}{l} \quad \frac{1}{l}\right] = \left[\frac{-1}{200} \quad \frac{1}{200}\right]$$

$$\{u\} = \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} 1.2 \\ 2.8 \end{Bmatrix}$$

$$\therefore \text{strain, } e = \begin{bmatrix} \frac{-1}{200} & \frac{1}{200} \end{bmatrix} \begin{Bmatrix} 1.2 \\ 2.8 \end{Bmatrix}$$

$$= \left[-\frac{1}{200} \times 1.2 + \frac{1}{200} \times 2.8 \right]$$

$$e = 8 \times 10^{-3}$$

iii) stress, $\sigma = E.e$

$$= 210 \times 10^{3} \times 8 \times 10^{-3}$$

$$= 1680 \text{ N/mm}^2$$

iv) strain Energy, $U = \frac{1}{2} \{u\}^T \{k\} \{u\}$

stiffness Matrix, $k = \frac{AE}{l} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

Area, $A = \frac{\pi}{4} d^2 = \frac{\pi}{4} \times 10^2 = 78.54 \text{ mm}^2$

$$\therefore k = \frac{78.54 \times 210 \times 10^{3}}{200} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\{u\}^T = \{u_1 \ u_2\} = \{1.2 \ 2.8\}$$

$$\therefore U = \frac{1}{2} \{1.2 \ 2.8\} \times 82467 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} 1.2 \\ 2.8 \end{Bmatrix}$$

$$= \frac{1}{2} \times 82467 \left[(1.2 - 2.8) \quad (-1.2 + 2.8) \right] \begin{Bmatrix} 1.2 \\ 2.8 \end{Bmatrix}$$

$$U = \frac{1}{2} \times 82467 \left[(-1.6 \times 1.2) + (1.6 \times 2.8) \right]$$

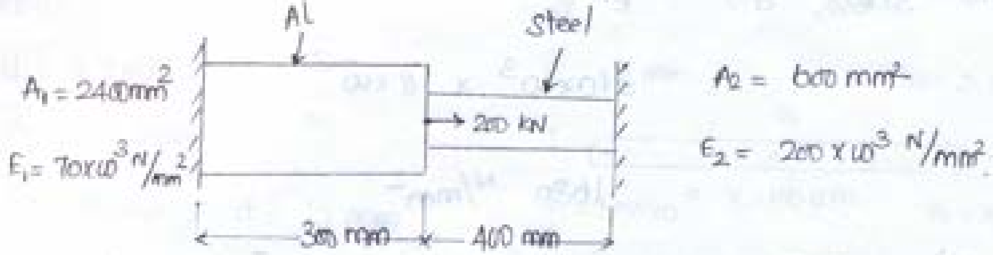
$$U = 105558 \text{ N.mm}$$

$$= 105.558 \text{ Nm.}$$

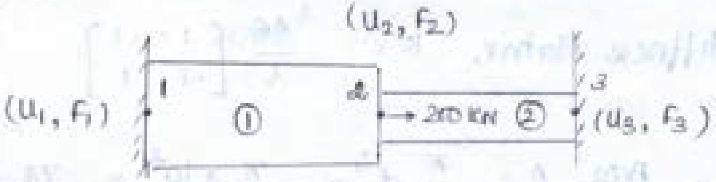
A Stepped bar is subjected to an axial load of 200 KN at the place of change of cross-Section and material as shown in fig.

Find a) Nodal displacements

b) Reaction forces c) Induced stress in each material.



solution:-

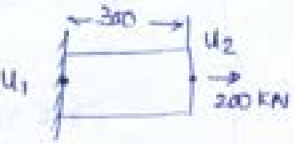


finite Element Equation for 1D two Noded bar element is given by.

$$\begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} = \frac{AE}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

For Element ①

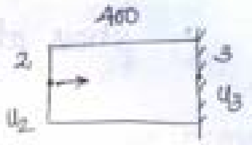
$$\begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} = \frac{2400 \times 70 \times 10^3}{300}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$



$$= 5.6 \times 10^5 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$\begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} = 10^5 \begin{bmatrix} \overset{a_{11}}{5.6} & \overset{a_{12}}{-5.6} \\ \overset{a_{21}}{-5.6} & \overset{a_{22}}{5.6} \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

For Element ②



$$\begin{Bmatrix} f_2 \\ f_3 \end{Bmatrix} = \frac{600 \times 200 \times 10^3}{400} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix}$$

$$= 3 \times 10^5 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix}$$

$$= 10^5 \begin{bmatrix} \overset{a_{22}}{3} & \overset{a_{23}}{-3} \\ \overset{a_{32}}{-3} & \overset{a_{33}}{3} \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix}$$

Assemble the finite Elements.

$$10^5 \begin{bmatrix} 5.6 & -5.6 & 0 \\ -5.6 & 8.6 & -3 \\ 0 & -3 & 3 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} f_1 \\ f_2 \\ f_3 \end{Bmatrix}$$

Applying boundary conditions:

Node 1 & 3 are fixed ; ∴ $u_1 = u_3 = 0$;

Self-weight is neglected ; $f_1 = f_3 = 0$;

Substituting in the assembled finite Elements.

$$10^5 \begin{bmatrix} 5.6 & -5.6 & 0 \\ -5.6 & 8.6 & -3 \\ 0 & -3 & 3 \end{bmatrix} \begin{Bmatrix} 0 \\ u_2 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 2 \times 10^5 \\ 0 \end{Bmatrix}$$

$$10^5 \left[8.6\, u_2 \right] = 2 \times 10^5$$

$$u_2 = 0.2325 \text{ mm};$$

Stress in each Element, stress, $\sigma = E \cdot e = E \cdot \frac{du}{dx}$

At Element ① $\sigma_1 = E_1 \cdot \frac{(u_2 - u_1)}{l_1}$ → $dx = l_1$

At Element ② $\sigma_2 = E_2 \cdot \frac{(u_3 - u_2)}{l}$

$\therefore$ stress at element ①, $\sigma_1 = 70\times10^3 \left(\frac{0.2325-0}{300} \right)$

$\sigma_1 = 54.25 \ N/mm^2$

at Element ②, $\sigma_2 = 200\times10^3 \left(\frac{0-0.2325}{400} \right)$

$= -116.25 \ N/mm^2$

(compressive stress)

Reaction force $\{R\}$

$$\{R\} = [K]\{u\} - \{F\}$$

$$\begin{Bmatrix} R_1 \\ R_2 \\ R_3 \end{Bmatrix} = 10^5 \begin{bmatrix} 5.6 & -5.6 & 0 \\ -5.6 & 8.6 & -3 \\ 0 & -3 & 3 \end{bmatrix} \begin{Bmatrix} 0 \\ 0.2325 \\ 0 \end{Bmatrix} - \begin{Bmatrix} 0 \\ 2\times10^5 \\ 0 \end{Bmatrix}$$

$$= 10^5 \begin{bmatrix} 0 - 5.6(0.2325)+0 \\ 0 \quad 8.6(0.2325)+0 \\ 0 \quad -3(0.2325)+0 \end{bmatrix} - \begin{Bmatrix} 0 \\ 2\times10^5 \\ 0 \end{Bmatrix}$$

$$= 10^5 \begin{bmatrix} -1.302 \\ 2 \\ -0.6975 \end{bmatrix} - \begin{Bmatrix} 0 \\ 2\times10^5 \\ 0 \end{Bmatrix}$$

$$\begin{Bmatrix} R_1 \\ R_2 \\ R_3 \end{Bmatrix} = \begin{Bmatrix} -1.302\times10^5 \\ 0 \\ -0.6975\times10^5 \end{Bmatrix}$$

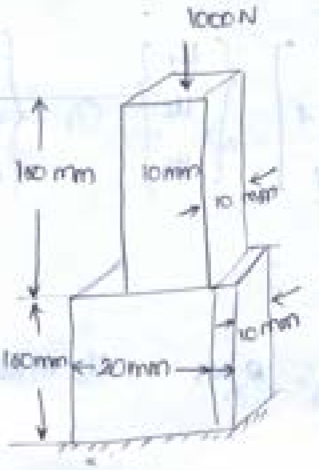
Reaction forces:-

$R_1 = -1.302\times10^5$ N ; $R_2 = 0$; $R_3 = -0.6975\times10^5$ N

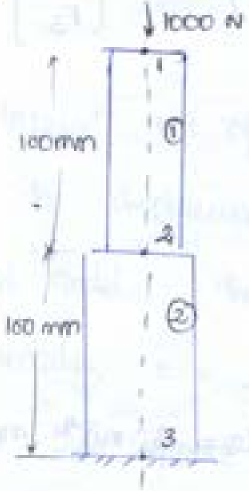
Verification: $R_1 + R_2 + R_3 = (-1.302 + 0 - 0.6975)\times10^5$

$= -200$ kN (Applied force)

For the column made of mild steel as ~~as~~ shown in fig. find a) Nodal displacements b) Reaction force at the support and c) stresses and strain displacements in element ①+②



The given column can be redrawn as element model containing two element with three nodes as shown in fig.



$A_1 = 10 \times 10 = 100 \text{ mm}^2$

$l_1 = 100 \text{ mm}$

$A_2 = 20 \times 10 = 200 \text{ mm}^2$

$l_2 = 100 \text{ mm}$

$E = E_1 = E_2 = 2 \times 10^5 \text{ N/mm}^2$ (mild steel).

Finite Element Equation,

$$\{K\}\{u\} = \{F\}$$

For Element ①

$$K_1 = \frac{A_1 E_1}{l_1}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 2 \times 10^5 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

For Element ②

$$K_2 = \frac{A_2 E_2}{L_2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 4 \times 10^5 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Stiffness matrix, $[K] = 10^5 \begin{bmatrix} 2 & -2 & 0 \\ -2 & 6 & -4 \\ 0 & -4 & 4 \end{bmatrix}$

Applying in the finite Element Equation.

$$10^5 \begin{bmatrix} 2 & -2 & 0 \\ -2 & 6 & -4 \\ 0 & -4 & 4 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix}$$

Applying Boundary condition:

$F_1 = 1000$ N ; $F_2 = 0$ N ; $u_3 = 0$;

(compr. force)

$$10^5 \begin{bmatrix} 2 & -2 & 0 \\ -2 & 6 & -4 \\ 0 & -4 & 4 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 1000 \\ 0 \\ F_3 \end{Bmatrix}$$

$$10^5 (2u_1 - 2(u_2) = 1000$$

$$10^5 (-2u_1 + 6u_2) = 0$$

Solving, we get

$u_1 = 75 \times 10^{-4}$ mm; $u_2 = 25 \times 10^{-4}$ mm;

and,

$$10^5 [0u_1 - 4u_2 + 0] = F_3$$

$$10^5 (-4 (25 \times 10^{-4})) = F_3$$

$$F_3 = -1000 \text{ N}$$

(Tensile force – Acting upwards).

stresses & strain in each element.

strain in element ① $= e_1 = \frac{du_1}{dx_1} = \frac{u_2 - u_1}{l_1}$

$$= \frac{(25 - 75) \times 10^{4}}{100} = -50 \times 10^{-6}$$

in element ② $= e_2 = \frac{u_3 - u_2}{l_2}$

$$= \frac{0 - (25 \times 10^{-4})}{100} = -25 \times 10^{-6}$$

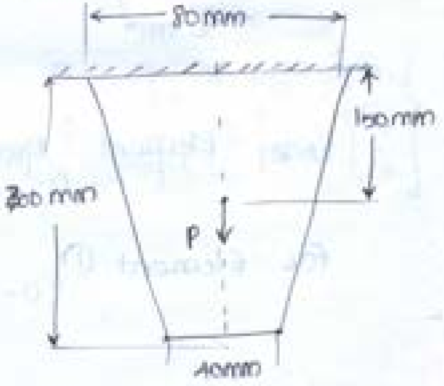
Stress in Element ① ; $\sigma_1 = E_1 . e_1 = 2 \times 10^5 (-50 \times 10^{-6})$

$$= -10 \text{ N/mm}^2$$

Element ②; $\sigma_2 = E_2 . e_2 = 2 \times 10^5 (-25 \times 10^{-6})$

$$= -5 \text{ N/mm}^2 .$$

For a tapered bar of uniform thickness $t = 10$mm as shown in fig. find the displacements at the nodes by forming into two element model. The bar has mass density $\rho = 7800$ kg/m³. Youngs Modulus, $E = 2 \times 10^5$ MN/m². Also determine the reaction force at the support if the bar is subjected to a self weight, $P = 1$ KN at it's centre



The area of c/s of the taper bar varies in lengthwise.

Inorder to simplify the Problem, the given tapered bar can be modelled into a Equivalent stepped bar having ② Elements.

For tappered bar.

Area at Node ① = width × thickness

$= 80 \times 10 = 800 \text{ mm}^2$

at node ③ $= 40 \times 10 = 400 \text{ mm}^2$

at Node ② $= \frac{\text{Area ① + Area ③}}{2} = \frac{800+400}{2} = 600 \text{ mm}^2$,

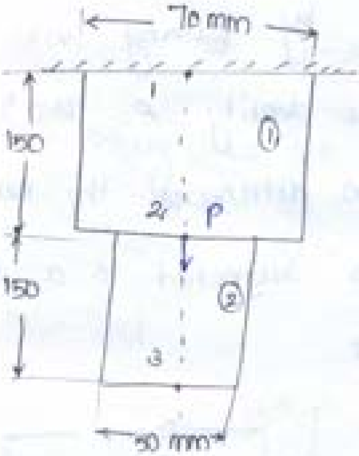
For stepped bar,

Area for Element ① $= \frac{\text{Area ① + Area ②}}{2}$

$= \frac{800+600}{2} = 700 \text{ mm}^2$ (70 mm width + 10 mm thickness)

Area for Element ② $= \frac{\text{Area ② + Area ③}}{2}$

$= \frac{600+400}{2} = 500 \text{ mm}^2$.



$P = 1000 \text{ N}$

$E_1 = E_2 = 2 \times 10^5 \text{ N/mm}^2$

$t = 10 \text{ mm}$

$\rho = 7800 \text{ kg/m}^3$

Finite Element Eqution: $\{K\}\{u\} = \{F\}$

For Element ①, $[K]^{①} = \frac{AE}{l_1}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

$= \frac{700 \times 2 \times 10^5}{150} = 0.94 \times 10^5 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

For Element ② $[k]^{②} = \frac{500 \times 2 \times 10^5}{150} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \underset{\times 10^5}{0.67} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

Stiffness Matrix, $[K] = \begin{bmatrix} 0.94 & -0.94 & 0 \\ -0.94 & 1.61 & -0.67 \\ 0 & -0.67 & 0.67 \end{bmatrix} \times 10^5$

Body force at element ①

$$W_1 = \rho g A L_1 = 7800 \times 10^{-9} \times 9.81 \times 700 \times 150$$

$$W_1 = 8.034 \text{ N}$$

at Element ②

$$W_2 = \rho g A_2 l_2 = 780 \times 10^{-9} \times 9.81 \times 500 \times 150$$

$$W_2 = 5.739 \text{ N}$$

Nodal forces,

At node 1, $F_1 = \frac{\text{Body force}}{2} = \frac{W_1}{2} = 4.017 \text{ N}$

At node 2, $F_2 = \frac{W_1}{2} + \frac{W_2}{2} + P = 4.017 + 2.87 + 1000$

$= 1006.887 \text{ N}$

At node 3, $F_3 = \frac{W_2}{2} = 2.87 \text{ N}$

Applying in Finite Element Equations,

$$10^5 \begin{bmatrix} 0.94 & -0.94 & 0 \\ -0.94 & 1.61 & -0.67 \\ 0 & 0.67 & 0.67 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix}$$

Applying Boundary conditions, $(u_1 = 0)$

$$10^5 \begin{bmatrix} 0.94 & -0.94 & 0 \\ -0.94 & 1.61 & -0.67 \\ 0 & 0.67 & 0.67 \end{bmatrix} \begin{Bmatrix} 0 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} F_1 \rightarrow 4.017\text{ N} \\ F_2 \rightarrow 1006.887\text{ N} \\ F_3 \rightarrow 2.87\text{ N} \end{Bmatrix}$$

$$10^5 (1.61 u_2 - 0.67 u_3) = 1006.887$$

$$10^5 (0.67 u_2 + 0.67 u_3) = 2.87$$

solving the two equations,

$$u_2 = 1082 \times 10^{-6} \text{ mm}$$

$$u_3 = 1086 \times 10^{-6} \text{ mm}.$$

Reaction force: $\{R\}$

$$\{R\} = [k][u] - \{F\}$$

$$\begin{Bmatrix} R_1 \\ R_2 \\ R_3 \end{Bmatrix} = 10^5 \begin{bmatrix} 0.94 & -0.94 & 0 \\ -0.94 & 1.61 & -0.67 \\ 0 & 0.67 & 0.67 \end{bmatrix} \begin{Bmatrix} 0 \\ 1082 \times 10^{-6} \\ 1086 \times 10^{-6} \end{Bmatrix} - \begin{Bmatrix} 4.017 \\ 1006.887 \\ 2.870 \end{Bmatrix}$$

$$R_1 = 10^5 \left[-0.94 \times 1082 \times 10^{-6} \right] - 4.017$$

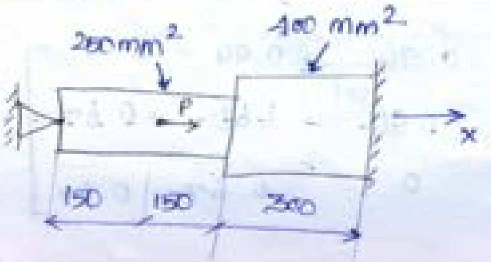
$$= -1013.8 \text{ N}.$$

Checking:- $R_1 + R_2 + R_3 = 4.017 + 1006.887 + 2.870$

$= 1013.8$ N (Acting upward).

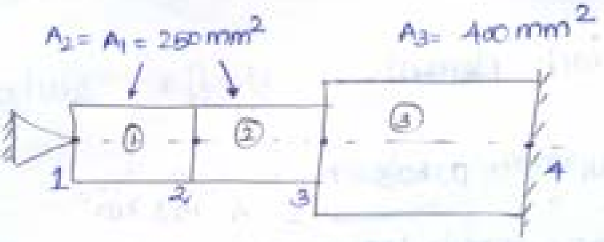
—x—

A stepped bar is subjected to an axial load of 300 kN as shown in fig. Find the nodal displacements, element stresses and support reactions.



$E = 200 \times 10^9$ N/m²

P = 300 KN



$$k_1 = \frac{A_1 E_1}{l_1}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} = \frac{250 \times 200 \times 10^3}{150}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} = 3.34 \times 10^5 \begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}$$

$$k_2 = 3.34 \times 10^5 \begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}$$

$$k_3 = 2.67 \times 10^5 \begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}$$

$$\therefore \text{ stiffness matrix, } [k] = 10^5 \begin{bmatrix}3.34 & -3.34 & 0 & 0\\ -3.34 & 6.68 & -3.34 & 0\\ 0 & -3.34 & 3.34+2.67 & -2.67\\ 0 & 0 & -2.67 & 2.67\end{bmatrix}$$

To finite Element Equation,

$$\{K\}\{u\} = \{F\}$$

$$10^5 \begin{bmatrix}3.34 & -3.34 & 0 & 0\\ -3.34 & 6.68 & -3.34 & 0\\ 0 & -3.34 & 6.01 & -2.67\\ 0 & 0 & -2.67 & 2.67\end{bmatrix}\begin{Bmatrix}u_1\\ u_2\\ u_3\\ u_4\end{Bmatrix} = \begin{Bmatrix}F_1\\ F_2\\ F_3\\ F_4\end{Bmatrix}$$

Applying Boundary condition:

$$u_1 = u_4 = 0 \;; \quad F_2 = 300 \times 10^3 \;; \quad F_3 = 0$$

By Gauss - Elimination Method,

$$10^5 (6.68\,u_2 - 3.34\,u_3) = 300 \times 10^3$$

$$10^5 (-3.34\,u_2 + 6.01\,u_3) = 0$$

Solving, we get

$$u_2 = 0.623\ mm;$$

$$u_3 = 0.346\ mm;$$

Strain & strain in each Element:

$$e_1 = \frac{du_1}{dx_1} = \frac{u_2 - u_1}{l_1} = \frac{0.623 - 0}{150} = 4.153 \times 10^{-3}$$

$$e_2 = \frac{u_3 - u_2}{l_2} = \frac{0.346 - 0.623}{150} = -1.847 \times 10^{-3}$$

$$e_3 = \frac{u_4 - u_3}{l_3} = \frac{0 - 0.346}{300} = -1.153 \times 10^{-3}$$

$$\sigma_1 = E \cdot e_1 = 200 \times 10^3 \times 4.153 \times 10^{-3} = 830.6 \text{ N/mm}^2 \text{ (Tensile)}$$

$$\sigma_2 = E \cdot e_2 = 200 \times 10^3 \times -1.847 \times 10^{-3} = -369.4 \text{ N/mm}^2 \text{ (compr)}$$

$$\sigma_3 = E \cdot e_3 = 2 \times 10^5 \times -1.153 \times 10^{-3} = -220.6 \text{ N/mm}^2 \text{ (compr)}$$

Support Reaction:-

from the Element Equation

$$10^5 \begin{bmatrix} 3.34 & -3.34 & 0 & 0 \\ -3.34 & 6.68 & -3.34 & 0 \\ 0 & -3.34 & 6.01 & -2.67 \\ 0 & 0 & -2.67 & 2.67 \end{bmatrix} \begin{Bmatrix} 0 \\ 0.623 \\ 0.346 \\ 0 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ 200 \times 10^3 \\ 0 \\ F_4 \end{Bmatrix}$$

$$10^5 \, (-3.34 \times 0.623) = F_1$$

$$10^5 \, (-2.67 \times 0.346) = F_4$$

$F_1 = -207.7$ kN ; $F_2 = -92.3$ kN

"−ve sign" → Opposite sign/direction.

Temperature Effects:

✓ when a machine member is loaded at room temperature, its dimensions changes and the corresponding stresses will be induced.

✓ If the ends are restricted, during the rise or fall of temperature, a stress called "thermal stress" is induced in the member (tensile (or) compressive in Nature).

✓ The strain due to change of temperature is known as thermal strain.

let, ΔT = change in temperature; (°C)

α = co-efficient of thermal expansion (mm/mm °c)

Then,

Thermal strain, $e = \alpha . \Delta T$

Thermal stress, $\sigma = E . e = E\alpha \Delta T$

Thermal force, $F = A . \sigma = AE\alpha \Delta T$

Nodal force vector due to rise in temperature,

$$[F] = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} = E . A . \alpha \Delta T \begin{Bmatrix} -1 \\ 1 \end{Bmatrix}$$

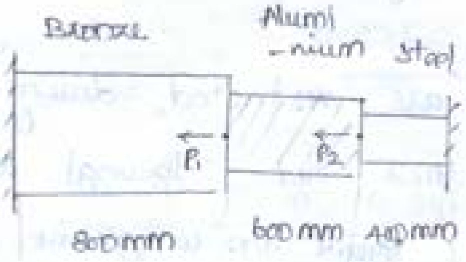
Thermal stress,

$$\{\sigma\} = E \frac{du}{dx} - E\alpha \Delta T$$

For Element ①, $\sigma_1 = E_1 \dfrac{(u_2 - u_1)}{l_1} - E_1 \alpha_1 \Delta T$

For Element ②, $\sigma_2 = E_2 \dfrac{(u_3 - u_2)}{l_2} - E_2 \alpha_2 \Delta T$

The stepped bar shown in fig. is subjected to an increase in temperature $\Delta T = 80°C$. Det. the displacements, Elemental stresses and Support reactions



$\alpha_B = 18.9 \times 10^{-6} /°C$

$\alpha_{Al} = 23 \times 10^{-6} /°C$

$\alpha_S = 11.7 \times 10^{-6} /°C$

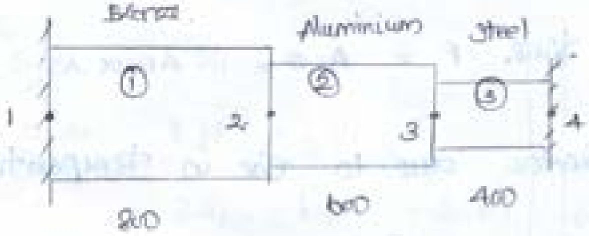
$P_1 = 60kN$; $P_2 = 75kN$.

	Bronze	Aluminium	Steel
A	$2400 mm^2$	$1200 mm^2$	$600 mm^2$
E	83 GPa	70 GPa	200 GPa

Given:-

$P_1 = -60 \times 10^3 N$; $P_2 = -75 \times 10^3 N$ (-ve Sign for loading acting on left)

$\Delta T = 80°C$



Solution:-

Finite Element Equation, $\{K\}\{u\} = \{F\}$

For 1-D 2 noded bar Element, $\frac{AE}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$

For Element ①

$$\frac{A_1 E_1}{l_1}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$$

$$\frac{2400 \times 83 \times 10^3}{800}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$$

$$10^3 \begin{bmatrix} 249 & -249 \\ -249 & 249 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix}$$

for Element ②

$$10^3 \begin{bmatrix} 140 & -140 \\ -140 & 140 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} f_2 \\ f_3 \end{Bmatrix}$$

For element ③

$$10^3 \begin{bmatrix} 300 & -300 \\ -300 & 300 \end{bmatrix} \begin{Bmatrix} u_3 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} f_3 \\ f_4 \end{Bmatrix}$$

∴ Stiffness matrix.

$$10^3 \begin{bmatrix} 249 & -249 & 0 & 0 \\ -249 & 249+140 & -140 & 0 \\ 0 & -140 & 140+300 & -300 \\ 0 & 0 & -300 & 300 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{Bmatrix}$$

Assembling the {F} matrix.

for Element ① $\begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix} = E_1 A_1 \alpha_1 \Delta T_1 \begin{Bmatrix} -1 \\ +1 \end{Bmatrix}$

$= 83 \times 10^3 \times 2400 \times 18.9 \times 10^{-6} \times 80 \begin{Bmatrix} -1 \\ 1 \end{Bmatrix}$

$= 10^3 \begin{Bmatrix} -301.19 \\ +301.19 \end{Bmatrix}$

for Element ② $\begin{Bmatrix} f_2 \\ f_3 \end{Bmatrix} = 10^3 \begin{Bmatrix} -154.56 \\ 154.56 \end{Bmatrix}$

for Element ③ $\begin{Bmatrix} f_3 \\ f_4 \end{Bmatrix} = 10^3 \begin{Bmatrix} -112.32 \\ 112.32 \end{Bmatrix}$

∴ {F} assembled matrix,

$$\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{Bmatrix} = 10^3 \begin{bmatrix} -301.19 \\ 301.19 - 154.56 \\ 154.56 - 112.32 \\ 112.32 \end{bmatrix} = 10^3 \begin{Bmatrix} -301.19 \\ 146.63 \\ 42.24 \\ 112.22 \end{Bmatrix}$$

Given axial load, $P_1 = -60 \times 10^3 N$ at Node 2

$P_2 = -75 \times 10^3 N$ at Node 3.

{F} matrix,

$$\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{Bmatrix} = 10^3 \begin{bmatrix} -301.19 \\ 146.63 - 60 \\ 42.24 - 75 \\ 112.32 \end{bmatrix} = 10^3 \begin{Bmatrix} -301.19 \\ 86.63 \\ -32.76 \\ 112.32 \end{Bmatrix}$$

Applying in the finite elemental Equation: {K}{u} = {F}

$$10^3 \begin{bmatrix} 249 & -249 & 0 & 0 \\ -249 & 389 & -140 & 0 \\ 0 & -140 & 440 & -300 \\ 0 & 0 & -300 & 300 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix} = 10^3 \begin{Bmatrix} -301.19 \\ 86.63 \\ -32.76 \\ 112.32 \end{Bmatrix}$$

Applying the Boundary condition:-

$u_1 = u_4 = 0;$

By Gauss, Elimination method.

$$10^3 \begin{bmatrix} 389 & -140 \\ -140 & 440 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = 10^3 \begin{Bmatrix} 86.63 \\ -32.76 \end{Bmatrix}$$

$$389u_2 - 140u_3 = 86.63$$

$$-140u_2 + 440u_3 = -32.76$$

Solving we get, ~~$u_2 =$~~ 0.2212 mm

$u_3 = -0.00345$ mm.

Thermal stress, $\sigma = E \cdot \frac{du}{dx} - E\alpha \Delta T$

For Element ①, $\sigma_1 = \frac{E_1 (u_2 - u_1)}{l_1} - E_1 \alpha_1 \Delta T$

$$= \frac{83 \times 10^3 (0.2212 - 0)}{800} - (83 \times 10^3 \times 18.9 \times 10^{-6} \times 80)$$

$\sigma_1 = -102.5455$ N/mm^2 (Compressive)

For Element ②, $\sigma_2 = \frac{E_2 (u_3 - u_2)}{l_2} - E_2 \alpha_2 \Delta T$

$\sigma_2 = -155.009$ N/mm^2 (compressive)

For Element ③, $\sigma_3 = \frac{E_3 (u_4 - u_3)}{l_3} - E_3 \alpha_3 \Delta T$

$\sigma_3 = -185.475$ N/mm^2 (compressive stress)

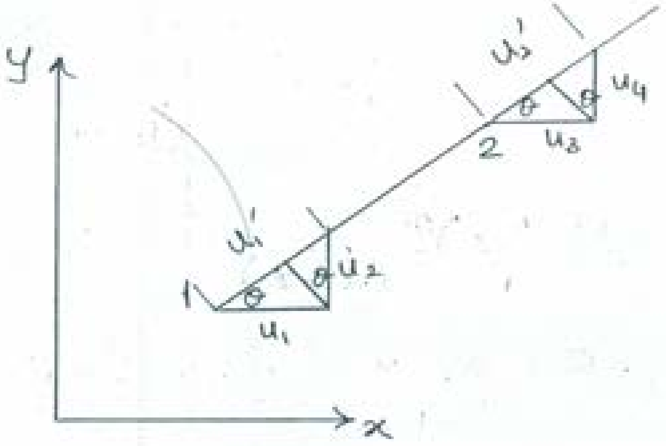
Reaction force, $\{R\} = [K][u] - \{F\}$

$$\begin{Bmatrix} R_1 \\ R_2 \\ R_3 \\ R_4 \end{Bmatrix} = 10^3 \begin{bmatrix} 249 & -249 & 0 & 0 \\ -249 & 389 & -140 & 0 \\ 0 & -140 & 440 & -300 \\ 0 & 0 & -300 & 300 \end{bmatrix} \begin{Bmatrix} 0 \\ 0.2212 \\ -0.00345 \\ 0 \end{Bmatrix} - 10^3 \begin{Bmatrix} -301.19 \\ 86.63 \\ -32.76 \\ 112.32 \end{Bmatrix}$$

Solving,

$$\begin{Bmatrix} R_1 \\ R_2 \\ R_3 \\ R_4 \end{Bmatrix} = 10^3 \begin{Bmatrix} 246.116 \\ 0 \\ 0 \\ -113.35 \end{Bmatrix}$$

Stiffness matrix for truss element



$$u_1' = u_1 \cos\theta + u_2 \sin\theta$$

$$u_2' = u_3 \cos\theta + u_4 \sin\theta$$

Let l & m be the direction cosines

$$l = \cos\theta \quad \& \quad m = \sin\theta$$

$$u_1' = u_1 l + u_2 m$$

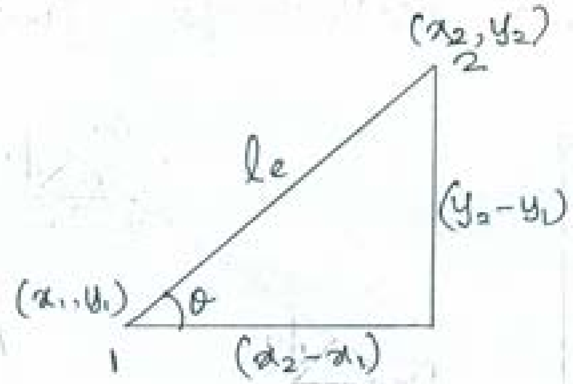
$$u_2' = u_3 l + u_4 m$$

The above equations can be written in matrix form as

$$\begin{Bmatrix} u_1' \\ u_2' \end{Bmatrix} = \begin{bmatrix} l & m & 0 & 0 \\ 0 & 0 & l & m \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix}$$

$$\{u'\} = [L]\{u\}$$

where $[L] = \begin{bmatrix} l & m & 0 & 0 \\ 0 & 0 & l & m \end{bmatrix}$ is called transformation matrix



$$l = \cos\theta = \frac{x_2 - x_1}{l_e} \quad ; \quad m = \sin\theta = \frac{y_2 - y_1}{l_e^{2}}$$

$$l_e = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Truss is also a one dimensional bar element.
(two noded)

$$[K'] = \frac{A_e E_e}{l_e} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

We known

Strain energy $U = 1/2 \{u'\}^T [K] \{u'\}$

We know $\{u'\} = [L]\{u\}$

$$U = 1/2 \left([L]\{u\}\right)^T [K'] [L] \{u\}$$

$$U = 1/2 \; [L]^T \{u\}^T [K'] [L] \{u\}$$

$$[K] = [L]^T [K'] [L]$$

$$U = 1/2 \; \{u\}^T \{u\} [K]$$

Element stiffness matrix in global coordinates

$$[K] = [L]^T [K'] [L]$$

$$[K] = \begin{bmatrix} l & 0 \\ m & 0 \\ 0 & l \\ 0 & m \end{bmatrix} \frac{A_e E_e}{l_e} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} l & m & 0 & 0 \\ 0 & 0 & l & m \end{bmatrix}$$

$$[K] = \frac{A_e E_e}{l_e} \begin{bmatrix} l & 0 \\ m & 0 \\ 0 & l \\ 0 & m \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} l & m & 0 & 0 \\ 0 & 0 & l & m \end{bmatrix}$$

$$= \frac{A_e E_e}{l_e} \begin{bmatrix} l & 0 \\ m & 0 \\ 0 & l \\ 0 & m \end{bmatrix} \begin{bmatrix} l-0 & m-0 & 0-l & 0-m \\ -l+0 & -m+0 & 0+l & 0+m \end{bmatrix}$$

$$= \frac{A_e E_e}{l_e} \begin{bmatrix} l & 0 \\ m & 0 \\ 0 & l \\ 0 & m \end{bmatrix} \begin{bmatrix} l & m & -l & -m \\ -l & -m & l & m \end{bmatrix}$$

$$= \frac{A_e E_e}{l_e} \begin{bmatrix} l^2-0 & lm-0 & -l^2+0 & -lm+0 \\ lm-0 & m^2-0 & -ml+0 & -m^2+0 \\ 0-l^2 & 0-lm & 0+l^2 & 0+lm \\ 0-lm & 0-m^2 & 0+lm & 0+m^2 \end{bmatrix}$$

$$[K] = \frac{A_e E_e}{l_e} \begin{bmatrix} l^2 & lm & -l^2 & -lm \\ lm & m^2 & -lm & -m^2 \\ -l^2 & -lm & l^2 & lm \\ -lm & -m^2 & lm & m^2 \end{bmatrix}$$

Finite Element Equation for 2 noded truss element

$$\{F\} = [K]\{u\}$$

$$\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{Bmatrix} = \frac{A_e E_e}{l_e} \begin{bmatrix} l^2 & lm & -l^2 & -lm \\ lm & m^2 & -lm & -m^2 \\ -l^2 & -lm & l^2 & lm \\ -lm & -m^2 & lm & m^2 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix}$$

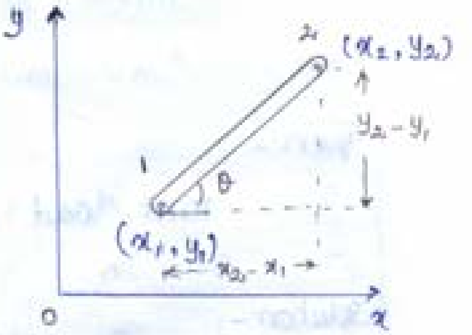
stiffness Matrix, $[K] = \frac{AE}{l} \begin{bmatrix} l^2 & lm & -l^2 & -lm \\ lm & m^2 & -lm & -m^2 \\ -l^2 & -lm & l^2 & lm \\ -lm & -m^2 & lm & m^2 \end{bmatrix}$

If the locations of the primary nodes 1, 2 are assumed as (x_1, y_1), $(x_2, y2)$ and length of truss element as l_e.

$$\cos\theta = \frac{x_2 - x_1}{l_e} = l_1$$

$$\sin\theta = \frac{y_2 - y_1}{l_e} = m_1$$

$$l_e = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$



Finite Element Equation for a two nodal truss Element

$$[F] = [K]\{u\}$$

$$\begin{Bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{Bmatrix} = \frac{AE}{l} \begin{bmatrix} l^2 & lm & -l^2 & -lm \\ lm & m^2 & -lm & m^2 \\ -l^2 & -lm & l^2 & lm \\ -lm & -m^2 & lm & m^2 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix}$$

stress formulation:-

stress induced in a truss, $\sigma = E.e = E.\frac{du}{dx} = \frac{E(u_2' - u_1')}{l}$

$$\sigma = \frac{E}{l}(u_2' - u_1') = \frac{E}{l}\begin{bmatrix} -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1' \\ u_2' \end{Bmatrix}$$

$$\{u'\} = [L]\{u\}$$

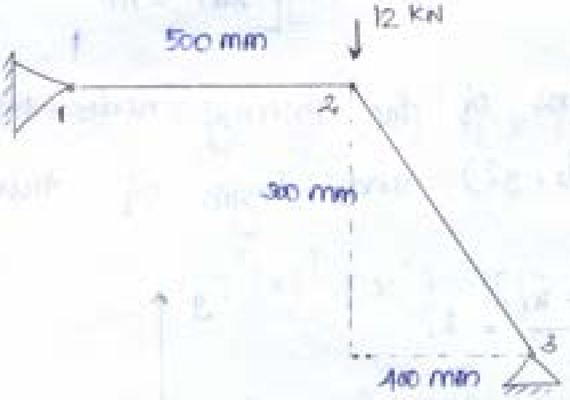
$$= \frac{E}{l}\begin{bmatrix} -1 & 1 \end{bmatrix}\{u'\}$$

$$= \frac{E}{l}\begin{bmatrix} -1 & 1 \end{bmatrix}[L][u] = \frac{E}{l}\begin{bmatrix} -1 & 1 \end{bmatrix}\begin{bmatrix} l & m & 0 & 0 \\ 0 & 0 & l & m \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix}$$

$$= \frac{E}{l}\begin{bmatrix} \cancel{l} & -m & l & m \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix}$$

Problems:-

For the two-bar truss as shown in fig. determine the displacements at node 2 and the stresses in both elements.



Take,

$E = 70$ GPa

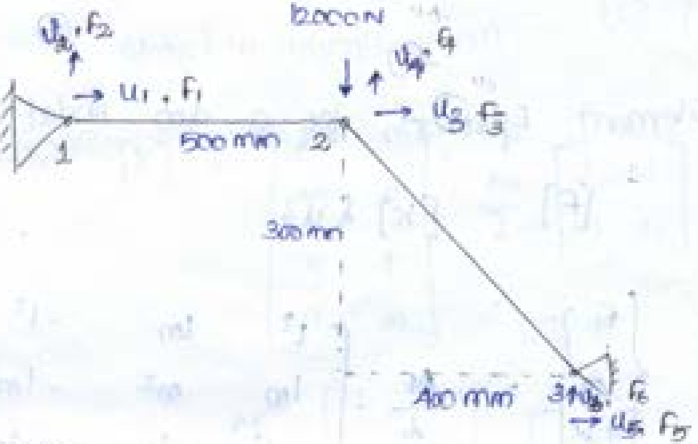
$A = 200\ mm^2$

Given:-

Point load at Node 2, $P_2 = 12$ kN (↓)

$= -12 \times 10^3$ N

Solution:-



co-ordinates

		x, y		x_2, y_2	
from Node 1	@ Node 1	(0, 0)	Node 1	(−500, 0)	from Node 2
	@ Node 2	(500, 0)	Node 2	x_1, y_1 (0, 0)	
	@ Node 3	(900, −300)	Node 3	(400, −300) x_3, y_3	

For Element ①, $E_1 = 70 \times 10^3\ N/mm^2$; $A_1 = 200\ mm^2$

$le_1 = 500$ mm

For Element ② $E_2 = 70 \times 10^3\ N/mm^2$; $A_2 = 200\ mm^2$

$$le_2 = \sqrt{400^2 + 300^2} = 500\ mm;$$

$$l_1 = \frac{x_2 - x_1}{le_1} = \frac{-500-0}{500} = -1; \quad m_1 = \frac{y_2 - y_1}{le_1} = 0;$$

$$l_2 = \frac{x_3 - x_1}{l_{e_2}} = \frac{400 - 0}{500} = 0.8$$

$$m_2 = \frac{y_3 - y_1}{l_{e_2}} = \frac{-300 - 0}{500} = -0.6.$$

Stiffness matrix [K]

for Element ①

$$[K]^{①} = \frac{A_1 E_1}{l_{e_1}} \begin{bmatrix} l_1^2 & l_1 m_1 & -l_1^2 & -l_1 m_1 \\ l_1 m_1 & m_1^2 & -l_1 m_1 & -m_1^2 \\ -l_1^2 & -l_1 m_1 & l_1^2 & l_1 m_1 \\ -l_1 m_1 & -m_1^2 & l_1 m_1 & m_1^2 \end{bmatrix}$$

$$= \frac{280 \times 70 \times 10^3}{500} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= 28 \times 10^3 \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \left[\begin{array}{cccc} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array} \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array}$$

for Element ②,

$$[K]^{②} = \frac{A_2 E_2}{l_{e_2}} \begin{bmatrix} l_2^2 & l_2 m_2 & -l_2^2 & -l_2 m_2 \\ l_2 m_2 & m_2^2 & -l_2 m_2 & -m_2^2 \\ -l_2^2 & -l_2 m_2 & l_2^2 & l_2 m_2 \\ -l_2 m_2 & -m_2^2 & l_2 m_2 & m_2^2 \end{bmatrix}$$

$$= 28 \times 10^3 \begin{array}{c} \begin{array}{cccc} 3 & 4 & 5 & 6 \end{array} \\ \left[\begin{array}{cccc} 0.64 & -0.48 & -0.64 & 0.48 \\ -0.48 & 0.36 & 0.48 & -0.36 \\ -0.64 & 0.48 & 0.64 & -0.48 \\ 0.48 & -0.36 & -0.48 & 0.36 \end{array}\right] \end{array} \begin{array}{c} 3 \\ 4 \\ 5 \\ 6 \end{array}$$

Assembly of the stiffness matrix [K]

$$[K] = 28\times10^3 \begin{array}{c} \\ \\ \end{array}\begin{array}{cccccc|c} 1 & 2 & 3 & 4 & 5 & 6 & \\ 1 & 0 & -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 \\ -1 & 0 & 1.64 & -0.48 & -0.64 & 0.48 & 3 \\ 0 & 0 & -0.48 & 0.36 & 0.48 & -0.36 & 4 \\ 0 & 0 & -0.64 & 0.48 & 0.64 & -0.48 & 5 \\ 0 & 0 & 0.48 & -0.36 & -0.48 & 0.36 & 6 \end{array}$$

Finite Element Equation, $[K][u] = [F]$

Applying Boundary condition,

$u_1 = u_2 = 0; \quad u_5 = u_6 = 0$

Self-weight is neglected, $F_1 = F_2 = F_3 = 0$

$F_5 = F_6 = 0$

$F_4 = -12\times10^3$ N;

Substituting in the finite Element Equations,

$$28\times10^3 \begin{bmatrix} 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1.64 & -0.48 & -0.64 & 0.48 \\ 0 & 0 & -0.48 & 0.36 & 0.48 & -0.36 \\ 0 & 0 & -0.64 & 0.48 & 0.64 & -0.48 \\ 0 & 0 & 0.48 & -0.36 & -0.48 & 0.36 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \\ F_6 \end{Bmatrix}$$

Since, $u_1 = u_2 = 0$ & $u_5 = u_6 = 0$; omit the 1st, 2nd ; 5th and 6th Row and columns.

$$28\times10^3\begin{bmatrix}1.64 & -0.48\\ -0.48 & 0.36\end{bmatrix}\begin{Bmatrix}u_3\\ u_4\end{Bmatrix}=\begin{Bmatrix}0\\ -12000\end{Bmatrix}$$

$$28\times10^3\ (\ 1.64\,u_3 - 0.48\,u_4) = 0$$

$$28\times10^3\ (\ -0.48\,u_3 + 0.36\,u_4 = -12000,$$

Solving, we get

$$u_3 = -0.57\ mm$$

$$u_4 = -1.96\ mm.$$

Stresses acting on the Element :-

Element ①
$$\sigma_1 = \frac{E_1}{L_{e_1}}\begin{bmatrix}-l_1 & -m_1 & l_1 & m_1\end{bmatrix}\begin{Bmatrix}u_1\\ u_2\\ u_3\\ u_4\end{Bmatrix}$$

Element ①
$$= \frac{70\times10^3}{500}\begin{bmatrix}1 & 0 & -1 & 0\end{bmatrix}\begin{Bmatrix}0\\ 0\\ -0.57\\ -1.96\end{Bmatrix}$$

$$= 140\ \begin{bmatrix}0.57\end{bmatrix}$$

$$= 79.8\ N/mm^2$$

Element ②,
$$\sigma_2 = \frac{E_2}{l_{e_2}}\begin{bmatrix}-l_2 & -m_2 & l_2 & m_2\end{bmatrix}\begin{Bmatrix}u_3\\ u_4\\ u_5\\ u_6\end{Bmatrix}$$

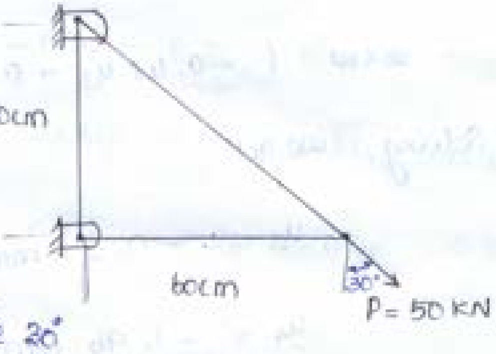
$$= 140\begin{bmatrix}-0.8 & 0.6 & 0.8 & -0.6\end{bmatrix}\begin{Bmatrix}-0.57\\ -1.96\\ 0\\ 0\end{Bmatrix}$$

$$= 140\ (\ (-0.8\times-0.57) + (0.6\times-1.96) + 0 + 0)$$

$$= -100.8\ N/mm^2 \quad \text{(compressive stress)}$$

Calculate nodal displacement and elemental stresses for the truss shown in fig. E = 70 GPa; cross-section area A = 2 cm² for all the truss members.

Given:-



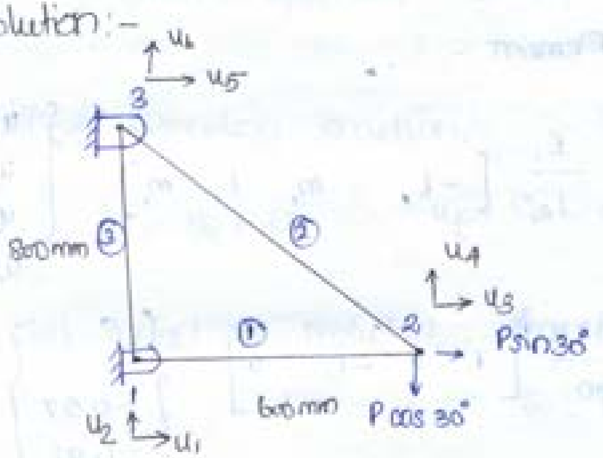
E = 70 GPa

= 70×10³ N/mm²

A = 200 mm³

Point load, P = 50×10³ N @ angle 20°

Solution:-



Node 1 $(\overset{x_1}{0}, \overset{y_1}{0})$

Node 2 $(\overset{x_2}{600}, \overset{y_2}{0})$

Node 3 $(\overset{x_3}{0}, \overset{y_3}{800})$

For Element ①

$$l_{e_1} = \sqrt{(x_2 - x_1)^2 + (y_2^2 - y_1)^2}$$

$$l_{e_1} = 600 \text{ mm}$$

$$l_1 = \frac{x_2 - x_1}{l_{e_1}} = \frac{600 - 0}{600} = 1;$$

$$m_1 = \frac{y_2 - y_1}{l_{e_1}} = 0;$$

Illy

For Element ②

$$l_{e_2} = 1000 \text{ mm}$$

$$l_2 = -0.6; \quad m_2 = 0.8$$

For Element ③ $l_{e_3} = 800$ mm

$$l_3 = 0; \quad m_3 = 1$$

Stiffness Matrix [K] for the truss Element.

$$[K] = \frac{AE}{l_e}\begin{bmatrix} l_1^2 & l_1 m_1 & -l_1^2 & -l_1 m_1 \\ l_1 m_1 & m_1^2 & -l_1 m_1 & -m_1^2 \\ -l_1^2 & -l_1 m_1 & l_1^2 & l_1 m_1 \\ -l_1 m_1 & -m_1^2 & l_1 m_1 & m_1^2 \end{bmatrix}$$

For Element ①

$$[K]_1 = 10^3 \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \left[\begin{array}{cccc} 23.3 & 0 & -23.3 & 0 \\ 0 & 0 & 0 & 0 \\ -23.3 & 0 & 23.3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array} \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array}$$

For Element ②

$$[K]_2 = 10^3 \begin{array}{c} \begin{array}{cccc} 3 & 4 & 5 & 6 \end{array} \\ \left[\begin{array}{cccc} 5 & -6.7 & -5 & 6.7 \\ -6.7 & 8.9 & 6.7 & -8.9 \\ -5 & 6.7 & 5 & -6.7 \\ 6.7 & -8.9 & -6.7 & 8.9 \end{array}\right] \end{array} \begin{array}{c} 3 \\ 4 \\ 5 \\ 6 \end{array}$$

For Element ③

$$[K]_3 = 10^3 \begin{array}{c} \begin{array}{cccc} 1 & 2 & 5 & 6 \end{array} \\ \left[\begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & 17.5 & 0 & -17.5 \\ 0 & 0 & 0 & 0 \\ 0 & -17.5 & 0 & 17.5 \end{array}\right] \end{array} \begin{array}{c} 1 \\ 2 \\ 5 \\ 6 \end{array}$$

Assembly of the stiffness matrix [K]

$$[K] = 10^3 \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \left[\begin{array}{cccccc} 23.3 & 0 & -23.3 & 0 & 0 & 0 \\ 0 & 17.5 & 0 & 0 & 0 & -17.5 \\ -23.3 & 0 & 28.3 & -6.7 & -5 & 6.7 \\ 0 & 0 & -6.7 & 8.9 & 6.7 & -8.9 \\ 0 & 0 & -5 & 6.7 & 5 & -6.7 \\ 0 & -17.5 & 6.7 & -8.9 & -6.7 & 26.4 \end{array}\right] \end{array} \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array}$$

Finite Element Equation: $[K]\{U\} = [F]$

Applying Boundary condition:-

$$u_1 = u_2 = 0 ; \quad u_5 = u_6 = 0$$

Self weight is not mentioned.

$$F_1 = F_2 = 0 ; \quad F_5 = F_6 = 0$$

$$F_3 = P \sin 30^\circ = 25 \times 10^3 \text{ N}$$

$$F_4 = -P \cos 30^\circ = -43.3 \times 10^3 \text{ N}$$

Substitute, the values in finite Element Equation,

Since, u_1, u_2, u_5, u_6 are 0 ;

Row 1, 2, 5 & 6 and columns are cancelled.

$$10^3 \begin{bmatrix} 28.37 & -6.7 \\ -6.7 & 8.9 \end{bmatrix} \begin{Bmatrix} u_3 \\ u_4 \end{Bmatrix} = 10^3 \begin{Bmatrix} 25 \\ -43.3 \end{Bmatrix}$$

Solving, $u_3 = -0.3256$ mm

$u_4 = -5.110$ mm.

Elemental stresses:-

$$\sigma = \frac{E}{l_e} \begin{bmatrix} -l & -m & l & m \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix}$$

for Element ①

$$\sigma_1 = \frac{70 \times 10^3}{600} \begin{bmatrix} -1 & 0 & 1 & 0 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ -0.3256 \\ -5.110 \end{Bmatrix}$$

$$= 116.66 \, (-0.3256)$$

$$= -37.98 \text{ N/mm}^2$$

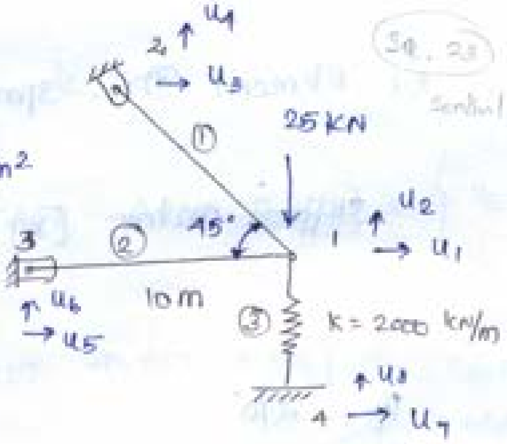
Illy,

$\sigma_2 = 272.51 \ N/mm^2$

$\sigma_3 = 0 \ N/mm^2.$

-x-

Consider a two-bar truss supported by a spring shown in fig. Both bars have $E = 210$ GPa & $A = 5 \times 10^{-4} m^2$. Bar one has a length of 5 m and bar 2 has a length of 10 m. Spring stiffness $K = 2$ kN/m. Det. the horizontal & vertical disp. at joint 1 & stress in each bar.



Given:-

$E = 210 \times 10^3 \ N/mm^2 = 210 \times 10^9 \ N/m^2$

$A = 5 \times 10^{-4} \ m^2$

$le_1 = 5 \ m \ ; \ le_2 = 10 m$

$K = 2 \ kN/m.$

Solution:-

For Element ① $\quad le_1 = 5 m;$ $\quad \theta = 135°$

$$l_1 = \cos\theta_1 = \frac{x_2 - x_1}{le_1} \ ;$$

$l_1 = \cos 135° = -0.707$ $\quad m_1 = \sin 135° = 0.707$

$$\text{stiffness matrix. } [K^1] = \frac{AE}{le_1} \begin{bmatrix} l_1^2 & l_1 m_1 & -l_1^2 & -l_1 m_1 \\ l_1 m_1 & m_1^2 & -l_1 m_1 & -m_1^2 \\ -l_1^2 & -l_1 m_1 & +l_1^2 & l_1 m_1 \\ -l_1 m_1 & m_1^2 & l_1 m_1 & m_1^2 \end{bmatrix}$$

$$= \frac{5 \times 10^{-4} \times 210 \times 10^9}{5} \begin{bmatrix} 0.5 & -0.5 & -0.5 & 0.5 \\ -0.5 & 0.5 & 0.5 & -0.5 \\ -0.5 & 0.5 & 0.5 & -0.5 \\ 0.5 & -0.5 & -0.5 & 0.5 \end{bmatrix}$$

$$[k]^{①} = 210\times10^5 \begin{array}{c} \\ \\ \\ \\ \\ \end{array}\begin{array}{cccc|c} 1 & 2 & 3 & 4 & \\ 0.5 & -0.5 & -0.5 & 0.5 & 1 \\ -0.5 & 0.5 & 0.5 & -0.5 & 2 \\ -0.5 & 0.5 & 0.5 & -0.5 & 3 \\ 0.5 & -0.5 & -0.5 & 0.5 & 4 \end{array}$$

For Element ②

$\theta_2 = 180°$; $l_2 = \cos\theta_2 = -1$; $m_2 = \sin\theta_2 = 0$

Ily.

Stiffness matrix, $[k]^{②} = 105\times10^5 \begin{array}{cccc|c} 1 & 2 & 5 & 6 & \\ +1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 2 \\ -1 & 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & 0 & 6 \end{array}$

For Element ③ – spring.

Stiffness matrix, $[k]^3 = \underset{\frac{AE}{l}}{\overset{\swarrow}{K}} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 \end{bmatrix}$

$\theta_3 = 270°$

$l_3 = \cos\theta_3 = 0$

$m_3 = \sin\theta_3 = 1$

$= 20\times10^5 \begin{array}{cccc|c} 1 & 2 & 7 & 8 & \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 & 2 \\ 0 & 0 & 0 & 0 & 7 \\ 0 & -1 & 0 & +1 & 8 \end{array}$

Assembly of the stiffness matrix [k]

$$K = 10^5 \begin{array}{cccccccc|c} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \\ 210 & 105 & -105 & 105 & 0 & 0 & 0 & 0 & 1 \\ -105 & 125 & 105 & -105 & 0 & 0 & 0 & -20 & 2 \\ -105 & 105 & 105 & -105 & 0 & 0 & 0 & 0 & 3 \\ 105 & -105 & -105 & 105 & 0 & 0 & 0 & 0 & 4 \\ -105 & 0 & 0 & 0 & 105 & 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 7 \\ 0 & -20 & 0 & 0 & 0 & 0 & 0 & -20 & 8 \end{array}$$

Applying Boundary condition in the finite Element Equation.

$$[K]\{u\} = [F]$$

$u_3 = u_4 = u_5 = u_6 = u_7 = u_8 = 0;$

All the forces will be zero except $F_2 = -25\times10^3$ N

Substituting the values we get,

$$10^5 \begin{bmatrix} 210 & -105 \\ -105 & 125 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} = 10^5 \begin{Bmatrix} 0 \\ -0.25 \end{Bmatrix}$$

Solving we get,

$u_1 = -1.724\times10^{-3}$ m

$u_2 = -3.448\times10^{-3}$ m.

Stresses acting on the truss element.

$$\sigma_1 = \frac{E}{l_{e_1}} \begin{bmatrix} 0.707 & -0.707 & -0.707 & 0.707 \end{bmatrix} \times \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix}$$

$$= \frac{210\times10^3}{5} \begin{bmatrix} 0.707 & -0.707 & -0.707 & 0.707 \end{bmatrix} \begin{bmatrix} -1.724\times10^{-3} \\ -3.448\times10^{-3} \\ 0 \\ 0 \end{bmatrix}$$

$\sigma_1 = 51.2\times10^6$ N/m^2

$$\sigma_2 = \frac{E}{l_{e_2}} \begin{bmatrix} 1 & 0 & -1 & 0 \end{bmatrix} \begin{array}{c c} \begin{Bmatrix} -1.724\times10^{-3} \\ -3.448\times10^{-3} \\ 0 \\ 0 \end{Bmatrix} & \begin{array}{c} u_1 \\ u_2 \\ u_5 \\ u_6 \end{array} \end{array}$$

$\sigma_2 = -36.2\times10^6$ N/m^2

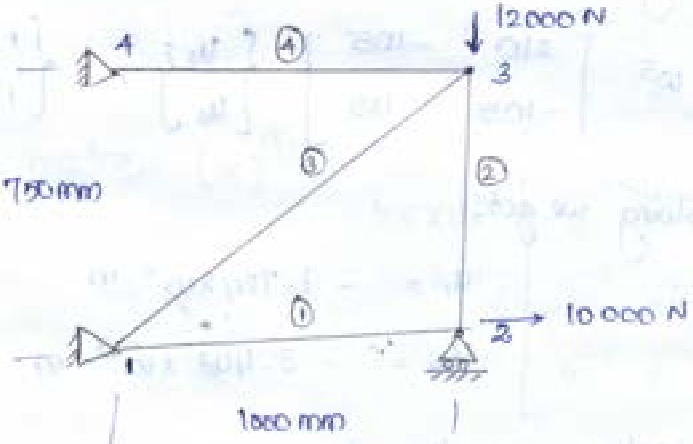
Consider a four bar truss as shown in fig.

Take, $E = 2\times10^5$ N/mm^2 and $A = 625$ mm^2 for all Elements.

Det. i) Element stiffness matrix for each element.

ii) Assemble the stiffness matrix [K]

iii) Solve for the nodal displacement.



Given:-

$E = 2\times10^5$ N/mm^2 ; $A = 625$ mm^2.

load at node 2 = 10×10^3 N ; at Node 3 = 12×10^3 N

Solution:-

	x_1, y_1
Node 1	(0, 0)
	x_2 y_2
2	(1000, 0)
	x_3 y_3
3	(1000, 750)
	x_4 y_4
4	(0, 750)

For element ① length $le_1 = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2}$

$$l_1 = \frac{x_2 - x_1}{le_1} \; ; \quad m = \frac{y_2 - y_1}{le_1}$$

$\therefore$ $le_1 = 1000$ mm;

$l_1 = 1$; $m_1 = 0$

Similarly,

For Element ②, l_{e_2} = 750 mm; $\lambda_2 = 0$; $m_2 = 1$

For Element ③, l_{e_3} = 1250 mm; $l_3 = 0.8$; $m_3 = 0.6$

For Element ④, l_{e_4} = 1000 mm; $\lambda_4 = 1$; $m_4 = 0$.

Stiffness matrix [K]

For, Element ①

$$[k]^{①} = \frac{AE}{l}\begin{bmatrix} l_1^2 & l_1m_1 & -l_1^2 & -l_1m_1 \\ l_1m_1 & m_1^2 & -l_1m_1 & -m_1^2 \\ -l_1^2 & -l_1m_1 & l_1^2 & l_1m_1 \\ -l_1m_1 & -m_1^2 & l_1m_1 & m_1^2 \end{bmatrix}$$

$$= 10^5 \begin{array}{c} \\ \left[\begin{array}{cccc} 1 & 2 & 3 & 4 \\ \hline 1.25 & 0 & -1.25 & 0 \\ 0 & 0 & 0 & 0 \\ -1.25 & 0 & 1.25 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}$$

||ly

For Element ②

$$(k)^{②} = 10^5 \begin{array}{c} \\ \left[\begin{array}{cccc} 3 & 4 & 5 & 6 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 1.67 & 0 & -1.67 \\ 0 & 0 & 0 & 0 \\ 0 & -1.67 & 0 & +1.67 \end{array}\right] \end{array} \begin{array}{c} \\ 3 \\ 4 \\ 5 \\ 6 \end{array}$$

For Element ③

$$[k]^{③} = 10^5 \begin{array}{c} \\ \left[\begin{array}{cccc} 1 & 2 & 5 & 6 \\ \hline 0.64 & 0.48 & -0.64 & -0.48 \\ 0.48 & 0.36 & -0.48 & -0.36 \\ -0.64 & -0.48 & 0.64 & 0.48 \\ -0.48 & -0.36 & 0.48 & 0.36 \end{array}\right] \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 5 \\ 6 \end{array}$$

For Element ④

$$[k]^{④} = 10^5 \begin{array}{c} \\ \left[\begin{array}{cccc} 5 & 6 & 7 & 8 \\ \hline 1.25 & 0 & -1.25 & 0 \\ 0 & 0 & 0 & 0 \\ -1.25 & 0 & 1.25 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array} \begin{array}{c} \\ 5 \\ 6 \\ 7 \\ 8 \end{array}$$

Assembly of the stiffness matrix, [K]

$$K = 10^5 \begin{bmatrix} 1.84 & 0.48 & -1.25 & 0 & -0.64 & -0.48 & 0 & 0 \\ 0.48 & 0.36 & 0 & 0 & -0.48 & -0.36 & 0 & 0 \\ -1.25 & 0 & 1.25 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1.67 & 0 & -1.67 & 0 & 0 \\ -0.64 & -0.48 & 0 & 0 & 1.84 & 0.48 & -1.25 & 0 \\ -0.48 & -0.36 & 0 & -1.67 & 0.48 & 2.02 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1.25 & 0 & 1.25 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

General finite Element Equation, $[K]\{u\} = [F]$

Applying Boundary conditions:

$u_1 = u_2 = 0;$ $u_7 = u_8 = 0;$ $u_4 = 0$

Self-wt is Neglected,

$F_1 = F_2 = F_4 = 0;$ $F_5 = F_7 = F_8 = 0;$

$F_6 = -12 \times 10^3 N;$ $F_3 = 10 \times 10^3 N.$

Since, $u_1 \mp u_2$, u_7, u_8, u_4 are zero.

The row & colums of 1, 2, 4, 7, 8 are cancelled.

Finite Element Equation;

$$10^5 \begin{bmatrix} 1.25 & 0 & 0 \\ 0 & 1.89 & 0.48 \\ 0 & 0.48 & 2.026 \end{bmatrix} \begin{Bmatrix} u_3 \\ u_5 \\ u_6 \end{Bmatrix} = 10^5 \begin{Bmatrix} 0.1 \\ 0 \\ -0.12 \end{Bmatrix}$$

Solving we get

$u_3 = 0.08$ mm

$u_5 = 0.016$ mm

$u_6 = -0.063$ mm

—x—.

Beams

Beam is a structural member having very long length compared to other two lateral dimensions such as width and height.

✓ Long horizontal members used in buildings, bridges and shafts supported in bearings are examples.

✓ Beams can be straight (or) curved.

✓ cross-section shape may be,

2 axes of symmetry (I-beams)
1 axes of symmetry (T-beams)
No axes of symmetry.

✓ Beam may be subjected to concentrated load and/or distributed load

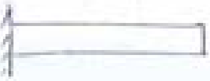
✓ Loads can act in one or more planes in the beam.

Types of Beams:-

Based on nature of support

1) cantilever Beam:

✓ one end is fixed & other end is free.

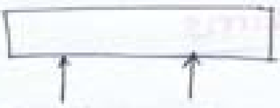


2) simply supported beam:

✓ Placed freely on the supports in both ends.

3) Over hanging beam:

✓ one or both ends extended beyond supports



4) fixed beam

✓ Both the ends are fixed

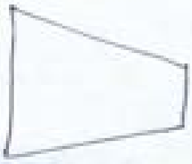
5) continuous beam.

✓ Beam is rested freely on more than 2 support.

Based on cross-section.

1) uniform cross section:

✓ same cross section throughout length.



varying cross section

2) varying cross-section:

cross section varying uniformly / stepped

stepped cross section

uniform cross section.

Shape function for linear beam element

Let deflection

$$w = a_0 + a_1 x + a_2 x^2 + a_3 x^3$$

$$= \langle 1 \; x \; x^2 \; x^3 \rangle \begin{Bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{Bmatrix}$$

Slope θ

$$\frac{dw}{dx} = 0 + a_1 + 2a_2 x + 3a_3 x^2$$

$$= \langle 0 \quad 1 \quad 2x \quad 3x^2 \rangle \begin{Bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{Bmatrix}$$

At $x=0$ deflection

$$w_i = a_0 + 0 + 0 + 0$$

$$\Longrightarrow a_0 = w_i$$

Slope

$$\theta_i = 0 + a_1 + 0 + 0$$

$$a_1 = \theta_i$$

At $x=l$ deflection

$$w_j = a_0 + a_1 l + a_2 l^2 + a_3 l^3$$

Sub $a_0 = w_i$ & $a_1 = \theta_i$

$$w_j = w_i + \theta_i l + a_2 l^2 + a_3 l^3$$

Slope

$$\theta_j = 0 + a_1 + 2a_2 l + 3a_3 l^2$$

Sub $a_1 = \theta_i$

$$\theta_j = 0 + \theta_i + a_2 l^2 + a_3 l^3$$

writing in the form of matrix

$$\begin{Bmatrix} w_i \\ \theta_i \\ w_j \\ \theta_j \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & l & l^2 & l^3 \\ 0 & 1 & 2l & 3l^2 \end{bmatrix} \begin{Bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{Bmatrix}$$

$$w_j - w_i = \cancel{w_i} + \theta_i l + a_2 l^2 + a_3 l^3 - \cancel{w_i}$$

$$w_j - w_i = \theta_i l + a_2 l^2 + a_3 l^3$$

$$w_j - w_i - \theta_i l = a_2 l^2 + a_3 l^3$$

$$\theta_j - \theta_i = \cancel{\theta_i} + 2a_2 l + 3a_3 l^2 - \cancel{\theta_i}$$

$$\theta_j - \theta_i = 2a_2 l + 3a_3 l^2$$

$$w_j - w_i - \theta_i l = a_2 l^2 + a_3 l^3$$
$$\theta_j - \theta_i = 2a_2 l + 3a_3 l^2$$

writing in matrix form

$$\begin{Bmatrix} w_j - w_i - \theta_i l \\ \theta_j - \theta_i \end{Bmatrix} = \begin{bmatrix} l^2 & l^3 \\ 2l & 3l^2 \end{bmatrix} \begin{Bmatrix} a_2 \\ a_3 \end{Bmatrix}$$

$$\begin{Bmatrix} a_2 \\ a_3 \end{Bmatrix} = \begin{bmatrix} l^2 & l^3 \\ 2l & 3l^2 \end{bmatrix}^{-1} \begin{Bmatrix} w_j - w_i - \theta_i l \\ \theta_j - \theta_i \end{Bmatrix}$$

$$= \frac{1}{l^4} \begin{bmatrix} 3l^2 & -l^3 \\ -2l & l^2 \end{bmatrix} \begin{Bmatrix} w_j - w_i - \theta_i l \\ \theta_j - \theta_i \end{Bmatrix}$$

$$\begin{Bmatrix} a_2 \\ a_3 \end{Bmatrix} = \begin{bmatrix} \frac{3\cancel{l^2}}{l^{\cancel{4}2}} & -\frac{\cancel{l^3}}{l^{\cancel{4}}} \\ -\frac{2\cancel{l}}{l^{\cancel{4}3}} & \frac{\cancel{l^2}}{l^{\cancel{4}2}} \end{bmatrix} \begin{Bmatrix} w_j - w_i - \theta_i l \\ \theta_j - \theta_i \end{Bmatrix}$$

$$a_2 = \frac{3}{l^2}(w_j - w_i - \theta_i l) - \frac{1}{l}(\theta_j - \theta_i)$$

$$= \frac{3}{l^2} w_j - \frac{3}{l^2} w_i - \frac{3}{l^2}\theta_i \cancel{l} - \frac{\theta_j}{l} + \frac{\theta_i}{l}$$

$$a_2 = \frac{3}{l^2} w_j - \frac{3}{l^2} w_i - \frac{2}{l}\theta_i - \frac{\theta_j}{l}$$

$$a_3 = -\frac{2}{l^3}(w_j - w_i - \theta_i l) + \frac{1}{l^2}(\theta_j - \theta_i)$$

$$= -\frac{2}{l^3} w_j + \frac{2}{l^3} w_i + \frac{2}{l^{\cancel{3}2}}\theta_i \cancel{l} + \frac{\theta_j}{l^2} - \frac{\theta_i}{l^2}$$

$$a_3 = -\frac{2}{l^3} w_j + \frac{2}{l^3} w_i + \frac{\theta_i}{l^2} + \frac{\theta_j}{l^2}$$

We know

$$a_0 = w_i$$

$$a_1 = \theta_i$$

$$a_2 = \frac{3}{l^2} w_j - \frac{3}{l^2} w_i - \frac{2\theta_i}{l} - \frac{\theta_j}{l}$$

$$a_3 = -\frac{2}{l^3} w_j + \frac{2}{l^3} w_i + \frac{\theta_i}{l^2} + \frac{\theta_j}{l^2}$$

Writing in matrix form

$$\begin{Bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -\frac{3}{l^2} & -2/l & 3/l & -1/l \\ 2/l^3 & 1/l^2 & -2/l^3 & 1/l^2 \end{bmatrix} \begin{Bmatrix} w_i \\ \theta_i \\ w_j \\ \theta_j \end{Bmatrix}$$

we know

$$w = \langle 1 \quad x \quad x^2 \quad x^3 \rangle \begin{Bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{Bmatrix}$$

$$w = \langle 1 \quad x \quad x^2 \quad x^3 \rangle \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -3/l^2 & -2/l & 3/l & -1/l \\ 2/l^3 & -1/l^2 & -2/l^3 & 1/l^2 \end{bmatrix} \begin{Bmatrix} w_i \\ \theta_i \\ w_j \\ \theta_j \end{Bmatrix}$$

$$\left(1 + 0x - \frac{3}{l^2}x^2 + \frac{2}{l^3}x^3\right)w_i$$

$$\left(0 + x - \frac{2}{l}x^2 + \frac{x^3}{l^2}\right)\theta_i$$

$$\left(0 + 0x + \frac{3x^2}{l} - \frac{2}{l^3}x^3\right)w_j$$

$$\left(0 + 0x - \frac{x^2}{l} + \frac{x^3}{l^2}\right)\theta_j$$

$$N_1 : 1 - \frac{3x^2}{l^2} + \frac{2x^3}{l^3}$$

$$N_2 = x - \frac{2x^2}{l} + \frac{x^3}{l^2}$$

$$N_3 = \frac{3x^2}{l} - \frac{2x^3}{l^3}$$

$$N_4 : \frac{-x^2}{l} + \frac{x^3}{l^2}$$

$$w = N_1 w_i + N_2 \theta_i + N_3 w_j + N_4 \theta_j$$

Shape function for linear beam element

$$N_1 = 1 - \frac{3x^2}{l^2} + \frac{2x^3}{l^3}$$

$$= \frac{1}{l^3}\left(l^3 - 3x^2 l + 2x^3\right)$$

$$N_1 = \frac{1}{l^3}\left(2x^3 - 3x^2 l + l^3\right)$$

$$N_2 = x - \frac{2x^2}{l} + \frac{x^3}{l^2}$$

$$= \frac{1}{l^3}\left(x l^3 - 2x^2 l^2 + x^3 l\right)$$

$$N_2 = \frac{1}{l^3}\left(x^3 l - 2x^2 l^2 + x l^3\right)$$

$$N_3 = \frac{3x^2}{l^2} - \frac{2x^3}{l^3}$$

$$= \frac{1}{l^3}\left(3x^2 l^{\cancel{2}} - 2x^3\right)$$

$$N_3 = \frac{1}{l^3}\left(-2x^3 + 3x^2 l\right)$$

$$N_4 = -\frac{x^2}{l} + \frac{x^3}{l^2}$$

$$= \frac{1}{l^3}\left(-x^2 l^2 + x^3 l\right)$$

$$N_4 = \frac{1}{l^3}\left(x^3 l - x^2 l^2\right)$$

$\therefore$

$$N_1 = \frac{1}{l^3}\left(2x^3 - 3x^2 l + l^3\right)$$

$$N_2 = \frac{1}{l^3}\left(x^3 l - 2x^2 l^2 + x l^3\right)$$

$$N_3 = \frac{1}{l^3}\left(-2x^3 + 3x^2 l\right)$$

[illegible] $x^2 l^2)$

Stiffness matrix for linear beam element

$$w(x) = N_1 w_i + N_2 \Phi_i + N_3 w_j + N_4 \theta_j$$

$$= \frac{1}{l^3}\left(2x^2 - 3x^2 l + l^3\right) w_i + \frac{1}{l^3}\left(x^3 l - 2x^2 l^2 + x l^3\right)\theta_i$$

$$+ \frac{1}{l^3}\left(-2x^3 + 3x^2 l\right) w_j + \frac{1}{l^3}\left(x^3 l - x^2 l^2\right)\theta_j$$

$$= \frac{2x^3}{l^3} w_i - \frac{3x^2 \cancel{l}}{l^{\cancel{3}2}} w_i + \frac{\cancel{l^3}}{\cancel{l^3}} w_i + \frac{x^3 \cancel{l}}{l^{\cancel{3}2}}\theta_i - \frac{2x^2 \cancel{l^2}}{l^{\cancel{3}}}\theta_i + \frac{x\cancel{l^3}}{\cancel{l^3}}\theta_i$$

$$- \frac{2x^3}{l^3} w_j + \frac{3x^2 \cancel{l}}{l^{\cancel{3}2}} w_j + \frac{x^3 \cancel{l}}{l^{\cancel{3}2}}\theta_j - \frac{x^2 \cancel{l}}{l^{\cancel{3}\cancel{2}}}\theta_j$$

$$= \frac{2x^3}{l^3} w_i - \frac{3x^2}{l^2} w_i + w_i + \frac{x^3}{l^2}\theta_i - \frac{2x^2}{l}\theta_i + x\theta_i$$

$$- \frac{2x^3}{l^3} w_j + \frac{3x^2}{l^2} w_j + \frac{x^3}{l^2}\theta_j - \frac{x^2}{l^{\cancel{2}}}\theta_j$$

$$= \frac{2x^3}{l^3} w_i + \frac{x^3}{l^2}\theta_i - \frac{2x^3}{l^3} w_j + \frac{x^3}{l^2}\theta_j$$

$$- \frac{3x^2}{l^2} w_i - \frac{2x^2}{l}\theta_i + \frac{3x^2}{l^2} w_j - \frac{x^2}{l^{\cancel{2}}}\theta_j$$

$$+ x\theta_i + w_i$$

$$w = \left[\frac{2}{l^3}(w_i - w_j) + \frac{1}{l^2}(\theta_i + \theta_j)\right] x^3$$

$$+ \left[\frac{-3}{l^2}(w_i - w_j) - \frac{1}{l}(+2\theta_i + \phi_j)\right] x^2$$

$$+ \phi_i x + w_i$$

$$\frac{dw(x)}{dx} = 3x^2\left[\frac{2}{l^3}(w_i - w_j) + \frac{1}{l^2}(\theta_i + \theta_j)\right]$$

$$+ 2x\left[\frac{-3}{l^2}(w_i - w_j) - \frac{1}{l}(2\theta_i + \theta_j)\right]$$

$$+ \theta_i$$

$$\frac{d^2w(x)}{dx} = 6x\left[\frac{2}{l^3}(w_i - w_j) + \frac{1}{l^2}(\theta_i + \theta_j)\right]$$

$$+ 2\left[\frac{-3}{l^2}(w_i - w_j) - \frac{1}{l}(2\theta_i + \theta_j)\right] \text{——} ①$$

$$\frac{d^3w(x)}{dx} = 6\left[\frac{2}{l^3}(w_i - w_j) + \frac{1}{l^2}(\theta_i + \theta_j)\right] \text{——} ②$$

Sub $x = 0$ in ①

$$\frac{d^2w(0)}{dx} = 0 + 2\left[\frac{-3}{l^2}(w_i - w_j) - \frac{1}{l}(2\theta_i + \theta_j)\right]$$

$$= \frac{-6}{l^2}(w_i - w_j) - \frac{2}{l}(2\theta_i + \theta_j)]$$

$$= \frac{1}{l^3}\left[-6l w_i + 6l w_j - 4l^2\theta_i - 2l^2\theta_j\right]$$

$$\frac{d^2w(0)}{dx} = \frac{1}{l^3}\left[-6l w_i - 4l^2\theta_i + 6l w_j - 2l^2\theta_j\right]$$

Sub $x = l$ in ①

$$\frac{d^2w(l)}{dx} = 6l\left[\frac{2}{l^3}(w_i - w_j) + \frac{1}{l^2}(\theta_i + \theta_j)\right]$$

$$+ 2\left[\frac{-3}{l^2}(w_i - w_j) - \frac{1}{l}(2\theta_i + \theta_j)\right]$$

$$= \frac{12\cancel{l}}{l^{\cancel{3}2}}(w_i - w_j) + \frac{6\cancel{l}}{l^{\cancel{2}}}(\theta_i + \theta_j)$$

$$- \frac{6}{l^2}(w_i - w_j) - \frac{2}{l}(2\theta_i + \theta_j)$$

$$= \frac{1}{l^3}\left(12 l w_i - 12 l w_j + 6l^2\theta_i + 6l^2\theta_j - 6l w_i + 6l w_j - 4l^2\theta_i - 2l^2\theta_j\right)$$

$$\frac{d^2 w(l)}{dx} = \frac{1}{l^3}\left(6 l w_i + 2l^2\theta_i - 6l w_j + 4 l^2\theta_j\right)$$

Put $x = 0$ in ②

$$\frac{d^3 w(0)}{dx} = 6\left[\frac{2}{l^3}(w_i - w_j) + \frac{1}{l^2}(\theta_i + \theta_j)\right]$$

$$= \frac{1}{l^3}\left[12 w_i - 12 w_j + 6l\theta_i + 6l\theta_j\right]$$

$$\frac{d^3 (w(0))}{dx} = \frac{1}{l^3}\left[12 w_i + 6l\theta_i - 12 w_j + 6l\theta_j\right]$$

Put $x = l$ in ②

$$\frac{d^3 w(l)}{dx} = \frac{1}{l^3}\left[12 w_i + 6l\theta_i - 12 w_j + 6l\theta_j\right].$$

Nodal force

$$F_1 = EI\frac{d^3 w(0)}{dx}$$

$$= \frac{EI}{l^3}\left[12 w_i + 6l\theta_i - 12 w_j + 6l\theta_j\right]$$

Bending Moment

$$M_1 = -EI\frac{d^2 w(0)}{dx^2}$$

$$= -\frac{EI}{l^3}\left[-6l w_i - 4l^2\theta_i + 6l w_j - 2l^2\theta_j\right]$$

$$= \frac{EI}{l^3}\left[6l w_i + 4l^2\theta_i - 6l w_j + 2l^2\theta_j\right]$$

Nodal force $F_2 = -EI \frac{d^3 w(l)}{dx^3}$

$$= \frac{-EI}{l^3}\left[12w_i + 6l\theta_i - 12w_j + 6l\theta_j\right]$$

$$= \frac{EI}{l^3}\left[-12w_i - 6l\theta_i + 12w_j - 6l\theta_j\right]$$

Bending Moment $M_2 = EI \frac{d^2 w(l)}{dx^2}$

$$= \frac{EI}{l^3}\left[6lw_i + 2l^2\theta_i - 6lw_j + 4l^2\theta_j\right]$$

$$F_1 = \frac{EI}{l^3}\left[12w_i + 6l\theta_i - 12w_j + 6l\theta_j\right]$$

$$M_1 = \frac{EI}{l^3}\left[6lw_i + 4l^2\theta_i - 6lw_j + 2l^2\theta_j\right]$$

$$F_2 = \frac{EI}{l^3}\left[-12w_i - 6l\theta_i + 12w_j - 6l\theta_j\right]$$

$$M_2 = \frac{EI}{l^3}\left[6lw_i + 2l^2\theta_i - 6lw_j + 4l^2\theta_j\right]$$

$$\begin{Bmatrix} F_1 \\ M_1 \\ F_2 \\ M_2 \end{Bmatrix} = \frac{EI}{l^3}\begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix}\begin{Bmatrix} w_i \\ \theta_i \\ w_j \\ \theta_j \end{Bmatrix}$$

Stiffness matrix

$$[K] = \frac{EI}{l^3}\begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix}$$

stiffness matrix, $[k] = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}$

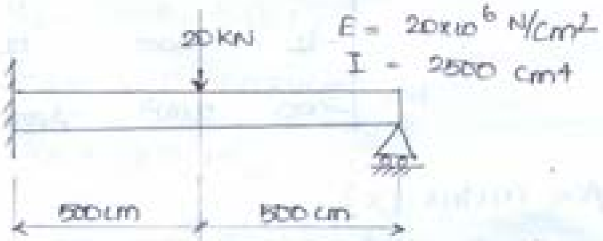
where,

E - Young's modulus

I - Moment of Inertia

L - length of the beam.

A beam is fixed at one end and supported by a roller at the other end, has a 20 KN concentrated load applied at the centre of the span, as shown in fig. calculate the deflection under the load and construct the shear force and bending moment diagram.



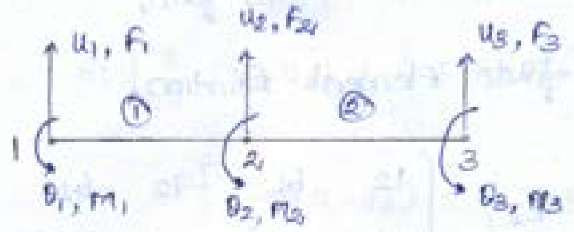
Solution:

let, the beam is divided into 2 elements ① & ② having nodes 1, 2 & 3.

u_1, θ_1, F_1, M_1 - deflection, slope, Shear force, Bending Moment at node 1.

u_2, θ_2, F_2, M_2 - " " " 2

u_3, θ_3, F_3, M_3 - " " " 3



$E = 20 \times 10^6 \ N/cm^2$

$I = 2500 \ cm^4$

For Element ①

stiffness matrix $[K]^{①} = \frac{E_1 I_1}{L_1^3} \begin{bmatrix} 12 & 6L & 12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}$

$$[K]^{①} = \frac{20 \times 10^6 \times 2500}{500^3} \begin{bmatrix} 12 & 3000 & -12 & 3000 \\ 3000 & 10^6 & -3000 & 5 \times 10^5 \\ -12 & -3000 & 12 & -3000 \\ 3000 & 5 \times 10^5 & -3000 & 10^6 \end{bmatrix}$$

$$= 400 \begin{array}{c} \begin{array}{cccc} u_1 & \theta_1 & u_2 & \theta_2 \end{array} \\ \left[\begin{array}{cccc} 12 & 3000 & -12 & 3000 \\ 3000 & 10^6 & -3000 & 5 \times 10^5 \\ -12 & -3000 & 12 & -3000 \\ 3000 & 5 \times 10^5 & -3000 & 10^6 \end{array}\right] \end{array} \begin{array}{c} u_1 \\ \theta_1 \\ u_2 \\ \theta_2 \end{array}$$

Illy for Element ②.

$$[K]^{②} = 400 \begin{array}{c} \begin{array}{cccc} u_2 & \theta_2 & u_3 & \theta_3 \end{array} \\ \left[\begin{array}{cccc} 12 & 3000 & -12 & 3000 \\ 3000 & 10^6 & -3000 & 5 \times 10^5 \\ -12 & -3000 & 12 & -3000 \\ 3000 & 5 \times 10^5 & -3000 & 10^6 \end{array}\right] \end{array} \begin{array}{c} u_2 \\ \theta_2 \\ u_3 \\ \theta_3 \end{array}$$

Global stiffness matrix [K]

$$K = 400 \begin{array}{c} \begin{array}{cccccc} u_1 & \theta_1 & u_2 & \theta_2 & u_3 & \theta_3 \end{array} \\ \left[\begin{array}{cccccc} 12 & 3000 & -12 & 3000 & 0 & 0 \\ 3000 & 10^6 & -3000 & 5 \times 10^5 & 0 & 0 \\ -12 & -3000 & 24 & 0 & -12 & 3000 \\ 3000 & 5 \times 10^5 & 0 & 2 \times 10^6 & -3000 & 5 \times 10^5 \\ 0 & 0 & -12 & -3000 & 12 & -3000 \\ 0 & 0 & 3000 & 5 \times 10^5 & -3000 & 10^6 \end{array}\right] \end{array} \begin{array}{c} u_1 \\ \theta_1 \\ u_2 \\ \theta_2 \\ u_3 \\ \theta_3 \end{array}$$

Finite Element Equation: $[K]\{u\} = \{F\}$

Applying Boundary condition

$u_1 = \theta_1 = 0; \quad u_3 = 0; \quad F_2 = -20\times10^3\ N;$

$M_2 = M_3 = 0;$

$$400\begin{bmatrix} 12 & 3000 & -12 & 3000 & 0 & 0 \\ 3000 & 10^6 & -3000 & 5\times10^5 & 0 & 0 \\ -12 & -3000 & 24 & 0 & -12 & 3000 \\ 3000 & 5\times10^5 & 0 & 2\times10^6 & -3000 & 5\times10^5 \\ 0 & 0 & -12 & -3000 & 12 & -3000 \\ 0 & 0 & 3000 & 5\times10^5 & -3000 & 10^6 \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ u_2 \\ \theta_2 \\ 0 \\ \theta_3 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ M_1 \\ -20\times10^3 \\ 0 \\ F_3 \\ 0 \end{Bmatrix}$$

Neglecting the 1, 2, 5th rows & columns

$$400\begin{bmatrix} 24 & 0 & 3000 \\ 0 & 2\times10^6 & 5\times10^5 \\ 3000 & 5\times10^5 & 10^6 \end{bmatrix}\begin{Bmatrix} u_2 \\ \theta_2 \\ \theta_3 \end{Bmatrix} = \begin{Bmatrix} -20\times10^3 \\ 0 \\ 0 \end{Bmatrix}$$

Solving we get,

$u_2 = -2.646\ cm$

$\theta_2 = -0.003125\ rad$

$\theta_3 = +0.0125\ rad.$

Shear force & Bending moment.

Shear force at node 1, 2 & 3

$F_1 = 400(-12u_2 + 3000\theta_2) = 13750\ N$

$F_2 = -20\times10^3\ N$ (Given)

$F_3 = 400(-12u_2 - 3000\theta_2 - 3000\theta_3) = 6250\ N.$

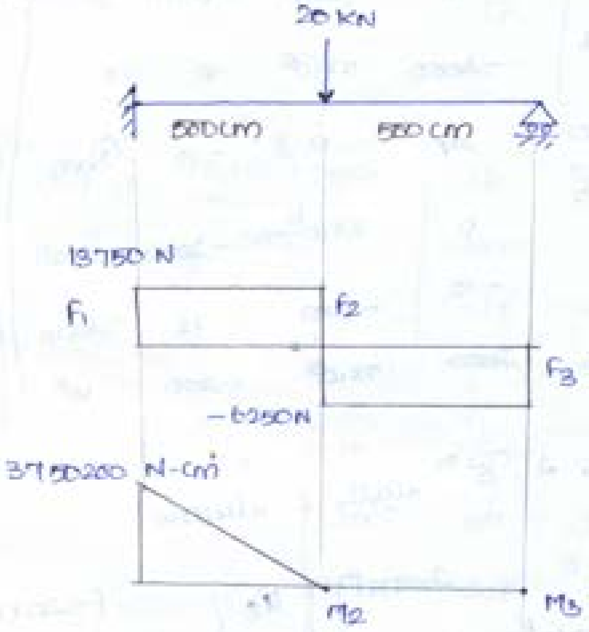
Bending moment at 1, 2, & 3.

$$M_1 = 400\ (-3000\ u_2 + 5\times10^5\ \theta_2)$$

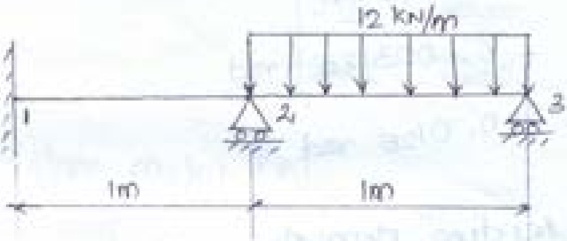
$$= 3750200\ \text{N-cm}$$

$$M_2 = M_3 = 0;$$

Diagram:-



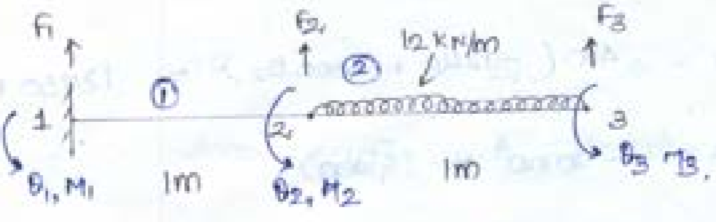
- For the beam loaded as shown in fig. determine the slopes at nodes 2 and 3. and the vertical deflection at the mid-point of the distributed load.



E = 200 GPa

$I = 4\times10^{-6}\ m^4$

let, the beam is divided into ② elements ①, ②



Let the nodal displacements.

for Element ①.

$$[K]^{①} = \frac{E_1 I_1}{l_1^3}\begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix}$$

$$\frac{E_1 I_1}{l_1^3} = \frac{200\times10^9 \times 4\times10^{-6}}{1^3} = 8\times10^5 \text{ N/m.}$$

$$[K]^{①} = 8\times10^5 \begin{array}{c} \begin{array}{cccc} u_1 & \theta_1 & u_2 & \theta_2 \end{array} \\ \begin{bmatrix} 12 & 6 & -12 & 6 \\ 6 & 4 & -6 & 2 \\ -12 & -6 & 12 & -6 \\ 6 & 2 & -6 & 4 \end{bmatrix} \end{array} \begin{array}{c} \\ u_1 \\ \theta_1 \\ u_2 \\ \theta_2 \end{array}$$

for Element ②

$$\frac{E_2 I_2}{l_2^3} = 8\times10^5 \text{ N/m.}$$

$$[K]^{②} = 8\times10^5 \begin{array}{c} \begin{array}{cccc} u_2 & \theta_2 & u_3 & \theta_3 \end{array} \\ \begin{bmatrix} 12 & 6 & -12 & 6 \\ 6 & 4 & -6 & 2 \\ -12 & -6 & 12 & -6 \\ 6 & 2 & -6 & 4 \end{bmatrix} \end{array} \begin{array}{c} \\ u_2 \\ \theta_2 \\ u_3 \\ \theta_3 \end{array}$$

Global stiffness matrix [K].

$$[K] = 8\times10^5 \begin{array}{c} \begin{array}{cccccc} u_1 & \theta_1 & u_2 & \theta_2 & u_3 & \theta_3 \end{array} \\ \begin{bmatrix} 12 & 6 & -12 & 6 & 0 & 0 \\ 6 & 4 & -6 & 2 & 0 & 0 \\ -12 & -6 & 24 & 0 & -12 & 6 \\ 6 & 2 & 0 & 8 & -6 & 2 \\ 0 & 0 & -12 & -6 & 12 & -6 \\ 0 & 0 & 6 & 2 & -6 & 4 \end{bmatrix} \end{array} \begin{array}{c} \\ u_1 \\ \theta_1 \\ u_2 \\ \theta_2 \\ u_3 \\ \theta_3 \end{array}$$

Nodal forces due to distributed load at node 2 & 3,

$$f_{2d} = f_{3d} = \frac{wl}{2} = \frac{12000\times1}{2} = 6000 \text{ N}$$

Nodal bending moments due to distributed load at 2 & 3.

$$M_{2d} = M_{3d} = \frac{wl^2}{12} = \frac{12000\times1^2}{12} = 1000 \text{ N-m.}$$

Finite Element Equation in matrix form, $[K]\{u\} = [F]$

$$8\times10^5 \begin{bmatrix} 12 & 6 & -12 & 6 & 0 & 0 \\ 6 & 4 & -6 & 2 & 0 & 0 \\ -12 & -6 & 24 & 0 & -12 & 6 \\ 6 & 2 & 0 & 8 & -6 & 2 \\ 0 & 0 & -12 & -6 & 12 & -6 \\ 0 & 0 & 6 & 2 & -6 & 4 \end{bmatrix} \begin{Bmatrix} u_1 \\ \theta_1 \\ u_2 \\ \theta_2 \\ u_3 \\ \theta_3 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ M_1 \\ F_2 + F_{2d} \\ M_2 + M_{2d} \\ F_3 + F_{3d} \\ M_3 + M_{3d} \end{Bmatrix}$$

Applying the boundary condition,

$u_1 = \theta_1 = 0; \quad u_2 = 0; \quad u_3 = 0;$

$F_{2d} = F_{3d} = -6000\,N; \quad M_{2d} = -1000\ Nm; \quad M_{3d} = 1000\ Nm$

$M_2 = M_3 = 0;$

$$8\times10^5 \begin{bmatrix} 12 & 6 & -12 & 6 & 0 & 0 \\ 6 & 4 & -6 & 2 & 0 & 0 \\ -12 & -6 & 24 & 0 & -12 & 6 \\ 6 & 2 & 0 & 8 & -6 & 2 \\ 0 & 0 & -12 & -6 & 12 & -6 \\ 0 & 0 & 6 & 2 & -6 & 4 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ \theta_2 \\ 0 \\ \theta_3 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ M_1 \\ F_2 - 600 \\ -1000 \\ F_3 - 6000 \\ 1000 \end{Bmatrix}$$

Since. $u_1 = u_2 = u_3 = 0; \quad \theta_1 = 0$

1st, 2nd, 3rd + 5th rows & column are cancelled.

$$8\times10^5 \begin{bmatrix} 8 & 2 \\ 2 & 4 \end{bmatrix} \begin{Bmatrix} \theta_2 \\ \theta_3 \end{Bmatrix} = \begin{Bmatrix} -1000 \\ 1000 \end{Bmatrix}$$

Solving, $\theta_2 = -2.67\times10^{-4}$ rad

$\theta_3 = 4.46\times10^{-4}$ rad.

To find the vertical deflection at the mid-pt of distributed load.

vertical deflection,

$$u = N_1 u_1 + N_2 \theta_1 + N_3 u_2 + N_4 \theta_2.$$

$$N_1 = 1 - \frac{3x^2}{l^2} + \frac{2x^3}{L^3} \quad ; \quad N_2 = x - \frac{2x^2}{l} + \frac{x^3}{L^2}$$

$$N_3 = \frac{3x^2}{L^2} - \frac{2x^3}{l^3} \quad ; \quad N_4 = \frac{-x^2}{L} + \frac{x^3}{L^2}$$

$$u_1 = u_2 = 0; \quad \theta_1 = -2.67 \times 10^{-4} \text{ rad}; \quad \theta_2 = 4.46 \times 10^{-4} \text{ rad};$$

$$u = N_1 (0) + N_2 \theta_1 + N_3 (0) + N_4 \theta_2$$

$N_2 \theta_1 \Rightarrow$ at $x = 0.5$ m (midpoint of Element ②)

$$N_2 \theta_1 = \left[0.5 - \frac{2(0.5^2)}{1} + \frac{0.5^3}{1^2} \right] (-2.67 \times 10^{-4})$$

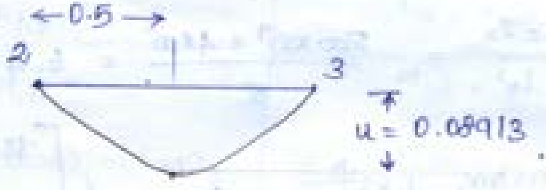
$$= -3.338 \times 10^{-5}$$

$$N_4 \theta_2 = \left[\frac{-0.5^2}{1} + \frac{0.5^3}{1^2} \right] (4.46 \times 10^{-4})$$

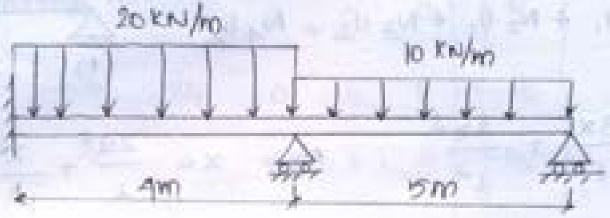
$$= -5.575 \times 10^{-5}$$

∴ vertical deflection, $u = 0 - (3.338 \times 10^{-5}) + 0 - (5.575 \times 10^{-5})$

$$u = -0.08913 \text{ mm}.$$



For the beam shown in fig. compute slope at the hinged support points. $E = 200\ GPa$; $I = 4\times10^{-6}\ m^4$. use two beam elements.

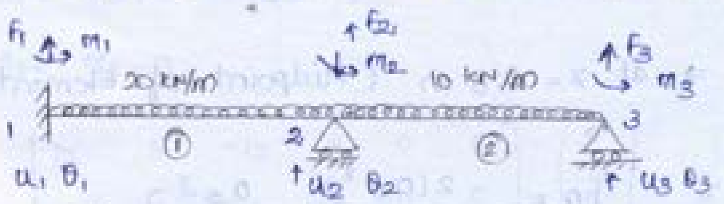


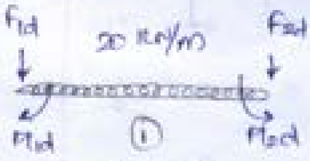
Solution:

let, beam is divided into two elements ① + ② having nodes 1, 2 & 3.

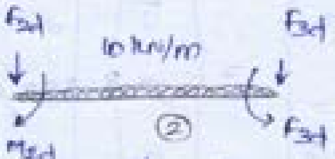
let, u_1, u_2, u_3 and $\theta_1, \theta_2, \theta_3$ displacements & slopes.

f_1, f_2, f_3 and M_1, M_2, M_3 – Shear force & Bending moments.





Element



Element

$$\text{Element stiffness matrix } [K] = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}$$

for Element ①,

$$\frac{E_1 I_1}{L_1^3} = \frac{200\times10^9 \times 4\times10^{-6}}{4} = 12.5\times10^3\ N/m$$

Element ②; $\frac{E_2 I_2}{L_2^3} = \frac{200\times10^9 \times 4\times10^{-6}}{5} = 6.4\times10^3\ N/m$

$$\text{stiffness matrix, } [K]^{①} = 12.5\times10^3 \begin{bmatrix} 12 & 24 & -12 & 24 \\ 24 & 64 & -24 & 32 \\ -12 & -24 & 12 & -24 \\ 24 & 32 & -24 & 64 \end{bmatrix}$$

$$[k]^{①} = 10^3 \begin{array}{c} \\ \\ \\ \\ \end{array}\begin{array}{cccc|c} u_1 & \theta_1 & u_2 & \theta_2 & \\ 150 & 300 & -150 & 300 & u_1 \\ 300 & 800 & -300 & 400 & \theta_1 \\ -150 & -300 & 150 & -300 & u_2 \\ 300 & 400 & -300 & 800 & \theta_2 \end{array}$$

lly

$$[k]^{②} = 10^3 \begin{array}{cccc|c} u_2 & \theta_2 & u_3 & \theta_3 & \\ 76.8 & 192 & -76.8 & 192 & u_2 \\ 192 & 640 & -192 & 320 & \theta_2 \\ -76.8 & -192 & 76.8 & -192 & u_3 \\ 192 & 320 & -192 & 640 & \theta_3 \end{array}$$

Global stiffness Matrix $[K]$:

$$[K] = 10^3 \begin{array}{cccccc|c} u_1 & \theta_1 & u_2 & \theta_2 & u_3 & \theta_3 & \\ 150 & 200 & -150 & 300 & 0 & 0 & u_1 \\ 300 & 800 & -300 & 400 & 0 & 0 & \theta_1 \\ -150 & -300 & 226.8 & -108 & -76.8 & 192 & u_2 \\ 300 & 400 & -108 & 1440 & -192 & 320 & \theta_2 \\ 0 & 0 & -76.8 & -192 & 76.8 & -192 & u_3 \\ 0 & 0 & 192 & 320 & -192 & 640 & \theta_3 \end{array}$$

Nodal forces & Moment

For Element ①,

$$F_{1d} = F_{2d} = \frac{W_1 l_1}{2} = \frac{20000 \times 4}{2} = 40 \times 10^3 \text{ N}$$

$$M_{1d} = M_{2d} = \frac{W_1 l_1^2}{12} = \frac{20000 \times 16}{12} = 26667 \text{ N-m}$$

For Element ②,

$$F_{2d} = F_{3d} = \frac{W_2 l_2}{2} = \frac{10000 \times 5}{2} = 25 \times 10^3 \text{ N}$$

$$M_{2d} = M_{3d} = \frac{W_2 l_2^2}{2} = \frac{10000 \times 25}{12} = 20833 \text{ Nm.}$$

Resultant Nodal forces & moments.

$$\{F\} = \begin{bmatrix} F_1 + F_{1d} \\ M_1 + M_{1d} \\ F_2 + (F_{2d})_1 + (F_{2d})_2 \\ M_2 + (M_{2d})_1 + (M_{2d})_2 \\ F_3 + F_{3d} \\ M_3 + M_{3d} \end{bmatrix} ;$$

$F_{1d} = F_{2d} = -40 \times 10^3$ N

$M_{1d} = -26667$ Nm

$(M_{2d})_1 = 26667$ Nm

$(F_{2d})_2 = F_{3d} = -25 \times 10^3$ N

$(M_{2d})_2 = -20833$ Nm

$M_{3d} = 20833$ Nm.

$$\therefore \{F\} = \begin{bmatrix} f_1 - 40\times10^3 \\ m_1 - 26667 \\ f_2 - 65000 \\ m_2 + 5834 \\ f_3 - 25\times10^3 \\ m_3 + 20833 \end{bmatrix}$$

Global finite Element Equation: $[K]\{u\} = [F]$

Applying Boundary conditions:

$u_1 = \theta_1 = 0$; $u_2 = u_3 = 0$; $F_2 = F_3 = 0$;

Eliminating, 1st, 2nd, 3rd, 4 5th rows + columns in the finite Element Quation.

$$10^3 \begin{bmatrix} 1440 & 320 \\ 320 & 640 \end{bmatrix} \begin{Bmatrix} \theta_2 \\ \theta_3 \end{Bmatrix} = \begin{Bmatrix} 5834 \\ 20833 \end{Bmatrix}$$

Solving;

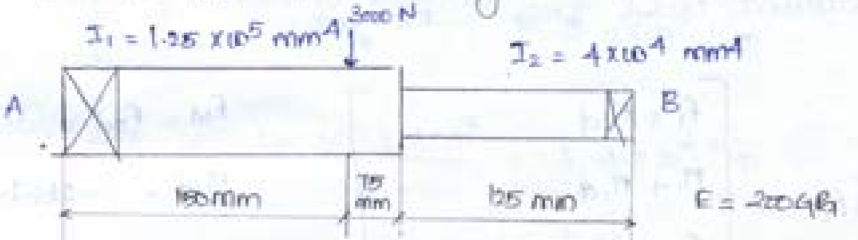
$$1440\,\theta_2 + 320\,\theta_3 = 5.834$$

$$320\,\theta_2 + 640\,\theta_3 = 20.833$$

$$\theta_2 = -3.58\times10^{-3} \text{ rad}$$

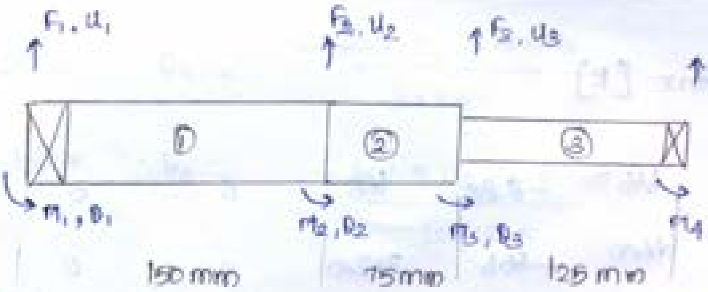
$$\theta_3 = 34.3\times10^{-3} \text{ rad.}$$

Find the deflection at the point of load and the slopes at the ends for the steel shaft which is Simply supported at the bearings A + B as shown in fig.



Solution:

The given beam element can be divided into ③ elements having nodes 1, 2, 3 + 4



Element stiffness matrix, $[K] = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}$

for Element ①, $\frac{E_1 I_1}{L_1^3} = \frac{2\times10^5 \times 1.25\times10^5}{150^3} = 0.74\times10^4$ N/mm

②, $\frac{E_2 I_2}{L^3} = \frac{2\times10^5\times1.25\times10^5}{75^3} = 5.93\times10^4$ N/mm

③, $\frac{E_3 I_3}{L_3^3} = \frac{2\times10^5\times1.56\times10^5}{125^3} = 0.41\times10^4$ N/mm.

stiffness matrix,

$$[k]_{①} = 0.74\times10^4 \begin{bmatrix} 12 & 900 & -12 & 900 \\ 900 & 9\times10^4 & -900 & 4.5\times10^4 \\ 12 & -900 & 12 & -900 \\ 900 & 4.5\times10^4 & -900 & 9\times10^4 \end{bmatrix}$$

$$= 10^4 \begin{array}{c} \begin{array}{cccc} u_1 & \theta_1 & u_2 & \theta_2 \end{array} \\ \begin{bmatrix} 8.88 & 666 & -8.88 & 666 \\ 666 & 66600 & -666 & 33300 \\ -8.88 & -666 & 8.88 & -666 \\ 666 & 33300 & -666 & 66600 \end{bmatrix} \end{array} \begin{array}{c} u_1 \\ \theta_1 \\ u_2 \\ \theta_2 \end{array}$$

My for,

$$[k]^{②} = 10^4 \begin{bmatrix} 71.16 & 2668.5 & -71.16 & 2668.5 \\ 2668.5 & 133425 & -2668.5 & 66712.5 \\ 71.16 & -2668.5 & 71.16 & -2668.5 \\ 2668.5 & 66712.5 & -2668.5 & 133425 \end{bmatrix}$$

$$[k]^{③} = 10^4 \begin{bmatrix} 4.92 & 307.5 & -4.92 & 307.5 \\ 307.5 & 25625 & -307.5 & 12812.5 \\ -4.92 & -307.5 & 4.92 & -307.5 \\ 307.5 & 12812.5 & -307.5 & 25625 \end{bmatrix}$$

Global stiffness matrix [K]

$$K = 10^4 \begin{bmatrix} 8.88 & 666 & -8.88 & 666 & 0 & 0 & 0 & 0 \\ 666 & 66600 & -666 & 33200 & 0 & 0 & 0 & 0 \\ -8.88 & -666 & 80.04 & 2002.5 & -71.16 & 2668.5 & 0 & 0 \\ 666 & 33300 & 2002.5 & 200025 & -2668.5 & 66712.5 & 0 & 0 \\ 0 & 0 & -71.16 & -2668.5 & 76.08 & -2361 & -4.92 & \\ 0 & 0 & 2668.5 & 66712.5 & -2361 & 159050 & -307.5 & \\ 0 & 0 & 0 & 0 & -4.92 & -307.5 & 4.92 & \\ 0 & 0 & 0 & 0 & 307.5 & 12812.5 & -307.5 & \end{bmatrix}$$

Global stiffness matrix. hand, finite Element Equation,

$$[K]\{u\} = [F]$$

Applying Boundary condition.

$u_1 = u_4 = 0$; $F_2 = -3000$ N; $F_3 = 0$;

$M_1 = M_2 = M_3 = M_4 = 0$;

Neglect 1st, 7th rows & columns in the finite Element Equations,

$$10^4 \begin{bmatrix} 66600 & -666 & 33300 & 0 & 0 & 0 \\ -666 & 80.04 & 2002.5 & -71.16 & 2668.5 & 0 \\ 33300 & 2002.5 & 200025 & -2668.5 & 66712.5 & 0 \\ 0 & -71.16 & -2668.5 & 76.08 & -2361 & 307.5 \\ 0 & 2668.5 & 66712.5 & -2361 & 159050 & 12812.5 \\ 0 & 0 & 0 & 307.5 & 12812.5 & 25625 \end{bmatrix} \begin{Bmatrix} \theta_1 \\ u_2 \\ \theta_2 \\ u_3 \\ \theta_3 \\ \theta_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -3000 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$66600\,\theta_1 - 666\,u_2 + 33300\,\theta_2 = 0 \quad \text{— (1)}$$

$$-666\,\theta_1 + 80.04\,u_2 + 2002.5\,\theta_2 - 71.16\,u_3 + 2668.5\,\theta_3 = -0.3 \quad \text{— ②}$$

$$33300\,\theta_1 + 2002.5\,u_2 + 200025\,\theta_2 - 2668.5\,u_3 + 66712.5\,\theta_3 = 0 \quad \text{— ③}$$

$$-71.16\,u_2 - 2668.5\,\theta_2 + 76.08\,u_3 - 2361\,\theta_3 + 307.5\,\theta_4 = 0 \quad \text{— ④}$$

$$2668.5\,u_2 + 66712.5\,\theta_2 - 2361\,u_3 + 159050\,\theta_3 + 12812.5\,\theta_4 = 0 \quad \text{— ⑤}$$

$$307.5\,u_3 + 12812.5\,\theta_3 + 25625\,\theta_4 = 0 \quad \text{— ⑥}$$

Equ(1) → $\theta_2 = \dfrac{-66600\,\theta_1 + 666\,u_2}{33300} = -2\theta_1 + 0.02\,u_2$

Equ (6) → $\theta_3 = \dfrac{-307.5\,u_3 - 25625\,\theta_4}{12812.5} = -0.024\,u_3 - 2\theta_4$

Applying in Equ (2), 3, 4, 5 we get,

Equ(2) → $-4671\,\theta_1 + 1201.\,u_2 - 135.2\,u_3 - 5337\,\theta_4 = -0.3$ — (7)

Equ(3) → $-266750\,\theta_1 + 6003\,u_2 - 4296.6\,u_3 - 133425\,\theta_4 = 0$ -(8)

Equ(4) → $5337\,\theta_1 - 124.5\,u_2 + 132.7\,u_3 + 5029.5\,\theta_4 = 0$ —(9)

Equ(5) → $-133425\,\theta_1 + 4002.8\,u_2 - 6178.2\,u_3 - 305288\,\theta_4 = 0$ —⑩

Equ(10) → $u_3 = \dfrac{133425\,\theta_1 - 4002.8\,u_2 + 305288}{-6178.2}$

$u_3 = -21.6\,\theta_1 + 0.65\,u_2 - 49.4\,\theta_4$

Substitute u_3 values in Equ (7), (8) & (9).

Equ(7) → $-1750.7\,\theta_1 + 32.2\,u_2 + 1341.9\,\theta_4 = -0.3$

(8) → $-274527\,\theta_1 + 3355.8\,u_2 + 77493\,\theta_4 = 0$

(9) → $2470.7\,\theta_1 - 38.2\,u_2 - 15259\,\theta_4 = 0$

solving, we get,

$\theta_1 = -0.0011$ rad

$\theta_4 = 0.0010$ rad

$u_2 = -0.112$ mm;

Shape function for quadratic element

Consider the quadratic equation

$$ax^2 + bx + c = 0$$

Rewriting

$$\phi = a_1 + a_2 x + a_3 x^2$$

$$\phi = < 1 \quad x \quad x^2 > \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix}$$

@ $x = 0$

$$\phi_1 = a_1 + a_2 (0) + a_3 (0)$$

$$\Longrightarrow \phi_1 = a_1$$

@ $x = l/2$

$$\phi_2 = a_1 + a_2 l/2 + a_3 \frac{l^2}{4} \qquad \Big| \; a_1 = \phi_1$$

$$\phi_2 = \phi_1 + a_2 l/2 + a_3 l^2/4$$

@ $x = l$

$$\phi_3 = a_1 + a_2 l + a_3 l^2$$

$$\phi_3 = \phi_1 + a_2 l + a_2 l^2$$

$\therefore$

$$\phi_1 = a_1 + a_2 0 + a_3 0$$

$$\phi_2 = a_1 + a_2 l/2 + a_3 \frac{l^2}{4}$$

$$\phi_3 = a_1 + a_2 l + a_3 l^2$$

$$\begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & \ell/2 & \ell^2/4 \\ 1 & \ell & \ell^2 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix}$$

$$\phi_2 - \phi_1 = \cancel{a_1} + a_2\frac{\ell}{2} + a_3\frac{\ell^2}{4} - \cancel{a_1} = a_2\frac{\ell}{2} + a_3\frac{\ell^2}{4}$$

$$\phi_3 - \phi_1 = \cancel{a_1} + a_2\ell + a_3\ell^2 - \cancel{a_1} = a_2\ell + a_3\ell^2$$

$$\begin{bmatrix} \phi_2 - \phi_1 \\ \phi_3 - \phi_1 \end{bmatrix} = \begin{bmatrix} \ell/2 & \ell^2/4 \\ \ell & \ell^2 \end{bmatrix} \begin{Bmatrix} a_2 \\ a_3 \end{Bmatrix}$$

$$\begin{Bmatrix} a_2 \\ a_3 \end{Bmatrix} = \begin{bmatrix} \ell/2 & \ell^2/4 \\ \ell & \ell^2 \end{bmatrix}^{-1} \begin{bmatrix} \phi_2 - \phi_1 \\ \phi_3 - \phi_1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} \operatorname{Adj} A$$

$$\begin{Bmatrix} a_2 \\ a_3 \end{Bmatrix} = \frac{1}{\ell^3/4} \begin{bmatrix} \ell^2 & -\ell^2/4 \\ -\ell & \ell/2 \end{bmatrix} \begin{bmatrix} \phi_2 - \phi_1 \\ \phi_3 - \phi_1 \end{bmatrix}$$

$$\begin{Bmatrix} a_2 \\ a_3 \end{Bmatrix} = \frac{4}{\ell^3} \begin{bmatrix} \ell^2 & -\ell^2/4 \\ -\ell & \ell/2 \end{bmatrix} \begin{bmatrix} \phi_2 - \phi_1 \\ \phi_3 - \phi_1 \end{bmatrix}$$

$$\begin{Bmatrix} a_2 \\ a_3 \end{Bmatrix} = \begin{bmatrix} 4/\ell & -1/\ell \\ -4/\ell^2 & 2/\ell^2 \end{bmatrix} \begin{Bmatrix} \phi_2 - \phi_1 \\ \phi_3 - \phi_1 \end{Bmatrix}$$

$$a_2 = \frac{4}{\ell}(\phi_2 - \phi_1) - \frac{1}{\ell}(\phi_3 - \phi_1)$$

$$= \frac{4}{\ell}\phi_2 - \frac{4}{\ell}\phi_1 - \frac{\phi_3}{\ell} + \frac{\phi_1}{\ell}$$

$$a_2 = -\frac{3}{\ell}\phi_1 + \frac{4}{\ell}\phi_2 - \frac{\phi_3}{\ell}$$

$$a_3 = -\frac{4}{\ell^2}(\phi_2 - \phi_1) + \frac{2}{\ell^2}(\phi_3 - \phi_1)$$

$$= -\frac{4}{\ell^2}\phi_2 + \frac{4}{\ell^2}\phi + \frac{2}{\ell^2}\phi_3 - \frac{2}{\ell^2}\phi_1$$

$$a_3 = \frac{2}{\ell^2}\phi_1 - \frac{4}{\ell^2}\phi_2 + \frac{2}{\ell^2}\phi_3$$

$$a_1 = \phi_1 + 0\phi_2 + 0\phi_3$$

$$a_2 = \frac{-3}{\ell}\phi_1 + \frac{4}{\ell}\phi_2 - \frac{\phi_3}{\ell}$$

$$a_3 = \frac{2}{\ell^2}\phi_1 - \frac{4}{\ell^2}\phi_2 + \frac{2}{\ell^2}\phi_3$$

$$\begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -3/\ell & 4/\ell & -1/\ell \\ 2/\ell^2 & -4/\ell^2 & 2/\ell^2 \end{bmatrix} \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{Bmatrix}$$

We know

$$\{\phi\} = \langle 1 \quad x \quad x^2 \rangle \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix}$$

$$\phi = \langle 1 \quad x \quad x^2 \rangle \begin{bmatrix} 1 & 0 & 0 \\ -3/l & 4/l & -1/l \\ 2/l^2 & -4/l^2 & 2/l^2 \end{bmatrix} \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{Bmatrix}$$

$$\phi = \left(1 - \frac{3}{l}x + \frac{2x^2}{l^2}\right)\phi_1$$

$$\phi = \left(\frac{4x}{l} - \frac{4x^2}{l^2}\right)\phi_2$$

$$\phi = \left(-\frac{x}{l} + \frac{2x^2}{l^2}\right)\phi_3$$

$$\phi = N_1\phi_1 + N_2\phi_2 + N_3\phi_3$$

where $N_1 = 1 - \frac{3x}{l} + \frac{2x^2}{l^2}$

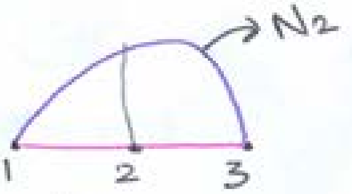
$N_2 = \frac{4x}{l} - \frac{4x^2}{l^2}$

$N_3 = -\frac{x}{l} + \frac{2x^2}{l^2}$

Properties of shape function

1. $(N_i)_j = 1$ if $i = j$ where j : node no
   $= 0$ if $i \neq j$ $\quad i$ = shape function

2. $\sum_{j=0}^{n} N_j = 1$



@ $x = 0$

$N_1 = 1 - \frac{3x}{l} + 2x = 1$

$N_2 = \frac{4x}{l} - \frac{4x^2}{l^2} = 0$

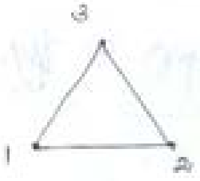
$N_3 = -\frac{x}{l} + \frac{2x^2}{l} = 0$

UNIT - III

TWO DIMENSIONAL SCALAR VARIABLE PROBLEMS.

Second Order 2D Equations involving Scalar variable functions – Variational formulation – Finite Element formulation – Triangular elements – shape functions and element matrices and vectors. Application to field Problems – Thermal problems – Torsion of Non-circular shafts – Quadrilateral elements – Higher Order Elements.

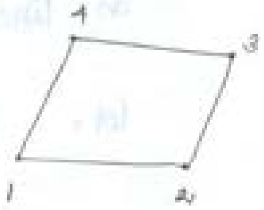
If the geometry and material properties of any element are described by two spatial coordinates, then the element is referred as two-dimensional element.



Triangle

Rectangle.

Quadrilateral



Parallelogram.

Ex: Plates under bi-axial loading.

Finite Element Modeling of 2D Elements.

Finite Element modeling means,

✓ discretizing the big sized element of irregular shape into many number of Calculatable regular shapes of Small size elements.

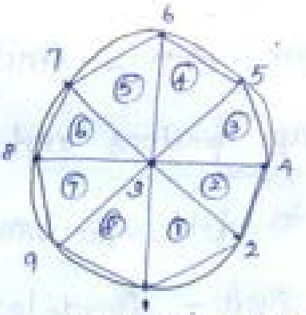
Possible shapes

✓ Triangle ✓ Rectangle ✓ Quadrilateral.

✓ Triangular element : simple 2D Element.

✓ other shaped elements : complex Element.


Before Discretization



After Discretization

8 - Elements ; 9 - Nodes;

Derivation of shape functions for 2D linear Element

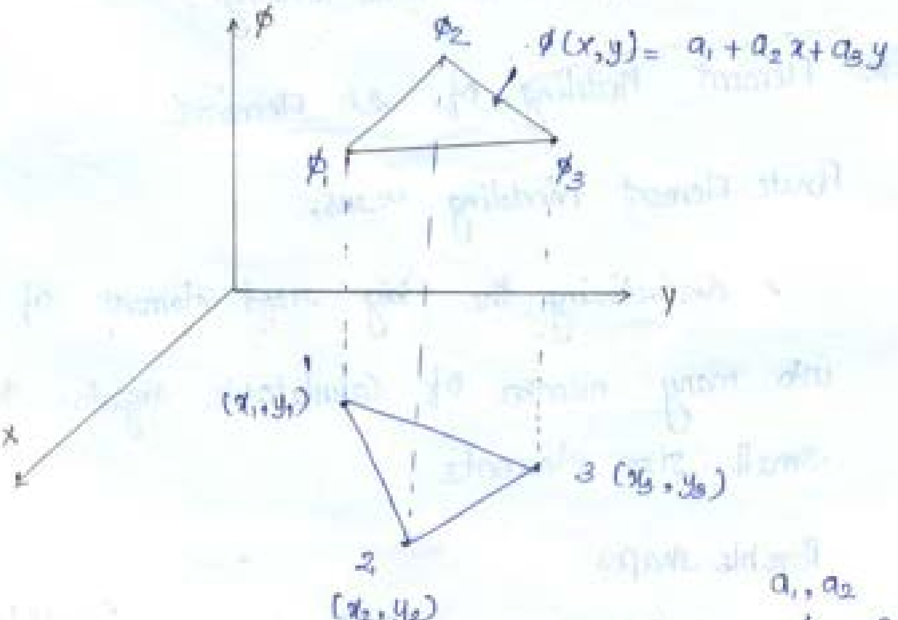
Consider a two dimensional straight sided triangular element with three corner nodes treated as linear (or) simple element as shown in fig.

let,

1, 2 & 3 are nodes;

(x_1, y_1) (x_2, y_2) (x_3, y_3) are global co-ordinates.

ϕ_1, ϕ_2 & ϕ_3 are field variables at nodes 1, 2 & 3.



$\phi(x, y) = a_1 + a_2 x + a_3 y$ —①

a_1, a_2 & a_3 → Polynomial co-eff.

then, $\phi_1 = a_1 + a_2 x_1 + a_3 y_1$

$\phi_2 = a_1 + a_2 x_2 + a_3 y_2$

$\phi_3 = a_1 + a_2 x_3 + a_3 y_3$

In matrix form,

$$\begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{Bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix}$$

then, $$\begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix}^{-1} \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{Bmatrix}$$

$$\{a\} = [D]^{-1} \{\phi\}$$

where,

$[D]$ = co-ordinate matrix = $\begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix}$

$[D]^{-1} = \dfrac{[C]^T}{|D|}$; $[C]$ = co-factor matrix

Now, $[C] = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$; $C_{ij} = (-1)^{i+j} |D_m|$
$\downarrow$
minor matrix

$C_{11} = (-1)^{1+1} \begin{vmatrix} x_2 & y_2 \\ x_3 & y_3 \end{vmatrix} = (x_2 y_3 - x_3 y_2)$

$||^{ly}$,

$C_{12} = -(y_3 - y_2) = y_2 - y_3$

$C_{13} = x_3 - x_2$;

$C_{21} = x_3 y_1 - x_1 y_3$;

$C_{22} = y_3 - y_1$;

$C_{23} = x_1 - x_3$;

$C_{31} = x_1 y_2 - x_2 y_1$;

$C_{32} = y_1 - y_2$;

$C_{33} = x_2 - x_1$;

co-factor matrix, $[C] = \begin{bmatrix} x_2 y_3 - x_3 y_2 & y_2 - y_3 & x_3 - x_2 \\ x_3 y_1 - x_1 y_3 & y_3 - y_1 & x_1 - x_3 \\ x_1 y_2 - x_2 y_1 & y_1 - y_2 & x_2 - x_1 \end{bmatrix}$

$$[C]^T = \begin{bmatrix} x_2 y_3 - x_3 y_2 & x_3 y_1 - x_1 y_3 & x_1 y_2 - x_2 y_1 \\ y_2 - y_3 & y_3 - y_1 & y_1 - y_2 \\ x_3 - x_2 & x_1 - x_3 & x_2 - x_1 \end{bmatrix}$$

Determinant of matrix, D

$$|D| = \begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix} = 1(x_2 y_3 - y_2 x_3) - x_1(y_3 - y_2) + y_1(x_3 - x_2)$$

$$= 2A$$

A → area of the triangle;

$$A = \frac{1}{2} \begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}$$

$$\therefore \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} = \frac{1}{2A} \begin{bmatrix} x_2 y_3 - x_3 y_2 & x_3 y_1 - y_3 x_1 & x_1 y_2 - x_2 y_1 \\ y_2 - y_3 & y_3 - y_1 & y_1 - y_2 \\ x_3 - x_2 & x_1 - x_3 & x_2 - x_1 \end{bmatrix} \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{Bmatrix}$$

$$= \frac{1}{2A} \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{bmatrix} \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{Bmatrix} \quad \text{—} ②$$

where,

$\alpha_1 = x_3 y_3 - x_3 y_2$; $\alpha_2 = x_3 y_1 - y_3 x_1$; $\alpha_3 = x_1 y_2 - x_2 y_1$;

$\beta_1 = y_2 - y_3$; $\beta_2 = y_3 - y_1$; $\beta_3 = y_1 - y_2$;

$\gamma_1 = x_3 - x_2$; $\gamma_2 = x_1 - x_3$; $\gamma_3 = x_2 - x_1$;

Equ(1) → $\phi(x,y) = a_1 + a_2 x + a_3 y$

In matrix form,

$$\phi(x,y) = \begin{bmatrix} 1 & x & y \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix}$$

Substituting Equ (2) in above equation,

$$\phi(x,y) = \begin{bmatrix} 1 & x & y \end{bmatrix} \frac{1}{2A} \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{bmatrix} \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{Bmatrix}$$

$$= \frac{1}{2A} \begin{bmatrix} \alpha_1 + \beta_1 x + \gamma_1 y & \alpha_2 + \beta_2 x + \gamma_2 y & \alpha_3 + \beta_3 x + \gamma_3 y \end{bmatrix} \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{Bmatrix}$$

$$= \begin{bmatrix} \frac{\alpha_1 + \beta_1 x + \gamma_1 y}{2A} & \frac{\alpha_2 + \beta_2 x + \gamma_2 y}{2A} & \frac{\alpha_3 + \beta_3 x + \gamma_3 y}{2A} \end{bmatrix} \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{Bmatrix}$$

$$\phi(x,y) = N_1 \phi_1 + N_2 \phi_2 + N_3 \phi_3$$

shape function,

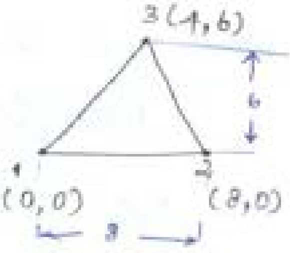
$$N_1 = \frac{\alpha_1 + \beta_1 x + \gamma_1 y}{2A}$$

$$N_2 = \frac{\alpha_2 + \beta_2 x + \gamma_2 y}{2A}$$

$$N_3 = \frac{\alpha_3 + \beta_3 x + \gamma_3 y}{2A}$$

To prove $|D| = \begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix} = 2A$

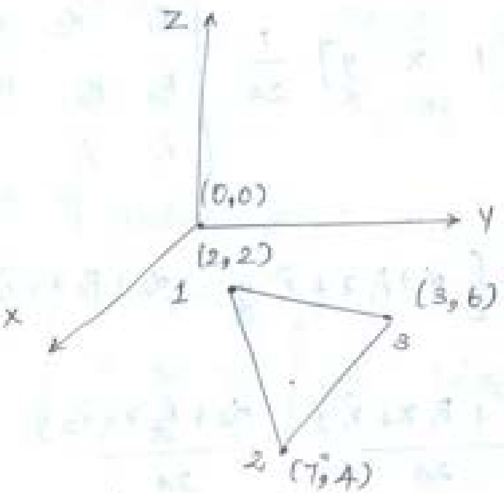
$A = \frac{1}{2} \times 8 \times 6 = 24$



$$|D| = \begin{vmatrix} 1 & 0 & 0 \\ 1 & 8 & 0 \\ 1 & 4 & 6 \end{vmatrix}$$

$= 48 = 2A$.

Derive the shape functions for a 3 noded triangular element as shown in fig. and using the same determine the temperature at the point P (4, 3), given that the temperatures at nodes 1, 2 and 3 are 75°C, 90°C & 60°C resp.



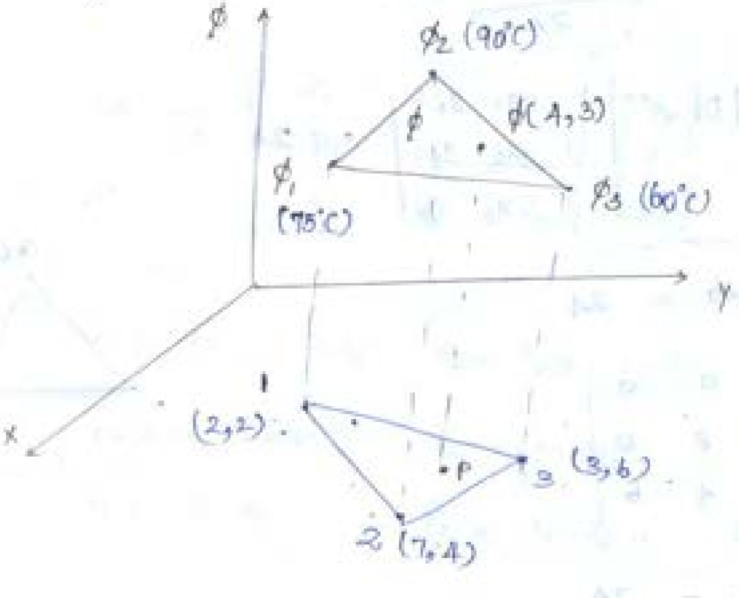
For, Node 1 $x_1 = 2$; $y_1 = 2$; $\phi_1 = 75°C$

2 $x_2 = 7$; $y_2 = 4$; $\phi_2 = 90°C$

3 $x_3 = 3$; $y_3 = 6$; $\phi_3 = 60°C$

solution:

To find the temperature at the required point, the given diagram is redrawn to shown nodal temperature.



Temperature at any point inside the 3 noded linear triangular element is given by,

$$\phi(x,y) = N_1 \phi_1 + N_2 \phi_2 + N_3 \phi_3$$

$$N_1 = \frac{1}{2A}(\alpha_1 + \beta_1 x + \gamma_1 y)$$

$$N_2 = \frac{1}{2A}(\alpha_2 + \beta_2 x + \gamma_2 y)$$

$$N_3 = \frac{1}{2A}(\alpha_3 + \beta_3 x + \gamma_3 y)$$

$$\alpha_1 = (x_2 y_3 - x_3 y_2) = (7\times6) - (3\times4) = 30$$

$$\alpha_2 = (x_3 y_1 - x_1 y_3) = (3\times2) - (2\times6) = -6$$

$$\alpha_3 = (x_1 y_2 - x_2 y_1) = (2\times4) - (7\times2) = -6$$

$$\beta_1 = y_2 - y_3 = 4-6 = -2$$

$$\beta_2 = y_3 - y_1 = 6-2 = 4$$

$$\beta_3 = y_1 - y_2 = 2-4 = -2$$

$$\gamma_1 = x_3 - x_2 = 3-7 = -4$$

$$\gamma_2 = x_1 - x_3 = 2-3 = -1$$

$$\gamma_3 = x_2 - x_1 = 7-2 = 5$$

$$A = \text{Area of the triangle} = \frac{1}{2}\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}$$

$$2A = \begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 2 \\ 1 & 7 & 4 \\ 1 & 3 & 6 \end{vmatrix}$$

$$= 1(42-12) - 2(6-4) + 2(3-7) = 30-4-8 = 18;$$

Substituting in the shape functions,

$$N_1 = \frac{1}{18}(30 - 2x - 4y)$$

$$N_2 = \frac{1}{18}(-6 + 4x - y)$$

$$N_3 = \frac{1}{18}(-6 - 2x + 5y)$$

Checking the correctness, $N_1 + N_2 + N_3 = 1$;

$$\frac{1}{18}(30 - 2x - 4y - 6 + 4x - y - 6 - 2x + 5y) = \frac{18}{18} = 1;$$

Temperature at the point P (4, 3) is given by,

$$\phi(4,3) = N_1\phi_1 + N_2\phi_2 + N_3\phi_3$$

$$= \frac{1}{18}(30 - 2x - 4y)\,75 + \frac{1}{18}(-6 + 4x - y)\,70 +$$

$$\frac{1}{18}(-6 - 2x + 5y)\,60$$

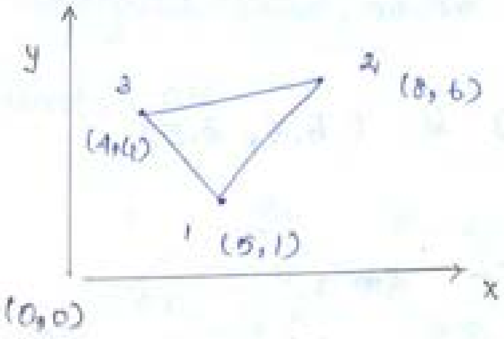
$$= \frac{1}{18}[750 + 630 + 60]$$

$$= \frac{1440}{18}$$

$$\phi(4,3) = 80^\circ C;$$

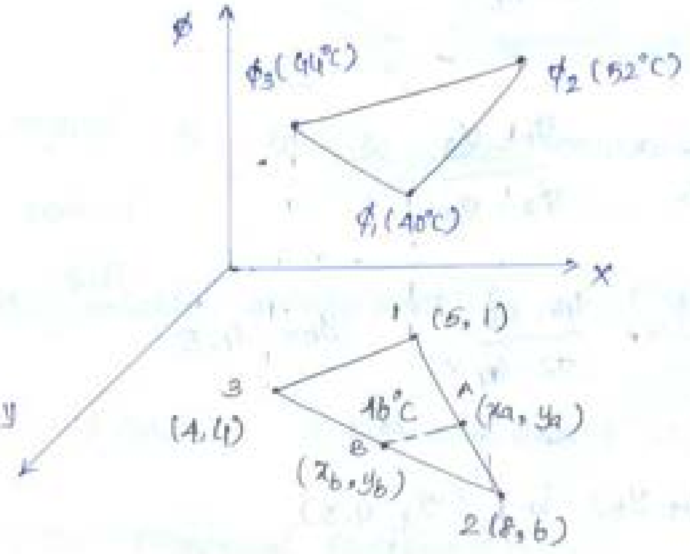
Temperature at the point P (4, 3) is 80°C;

For a 3 noded linear triangular element as shown in fig. determine isotherm corresponding to 46°C; The temperatures at nodes 1, 2 and 3 are 40°C; 52°C + 44°C respectively.



Solution:-

The given diagram is redrawn to show the temperature values at the nodes.



Let, (x_a, y_a) + (x_b, y_b) are the co-ordinates of the points A and B at the sides 1-2 and 2-3 having temperature of 46°c.

To find (x_a, y_a) 1 to 2.

$$\frac{x_a - x_1}{x_2 - x_1} = \frac{\phi_a - \phi_1}{\phi_2 - \phi_1}$$

$$\frac{x_a - 5}{8 - 5} = \frac{46 - 40}{52 - 40} ; \quad x_a = 6.5;$$

$$\frac{y_a - y_1}{y_2 - y_1} = \frac{\phi_a - \phi_1}{\phi_2 - \phi_1}$$

$$\frac{y_a - 1}{6 - 1} = \frac{46 - 40}{52 - 40} ; \quad y_a = 3.5.$$

Hence, (x_a, y_a) is $(6.5, 3.5)$.

To find (x_b, y_b) :- 2 to 3.

$$\frac{x_b - x_3}{x_2 - x_3} = \frac{\phi_b - \phi_3}{\phi_2 - \phi_3}$$

$$\frac{x_b - 4}{2 - 4} = \frac{46 - 44}{52 - 44} ; \quad x_b = 5.$$

$$\frac{y_b - y_3}{y_2 - y_3} = \frac{\phi_b - \phi_3}{\phi_2 - \phi_3}$$

$$\frac{y_b - 4}{6 - 4} = \frac{46 - 44}{52 - 44} ; \quad y_b = 4.5.$$

Hence, (x_b, y_b) is $(5, 4.5)$.

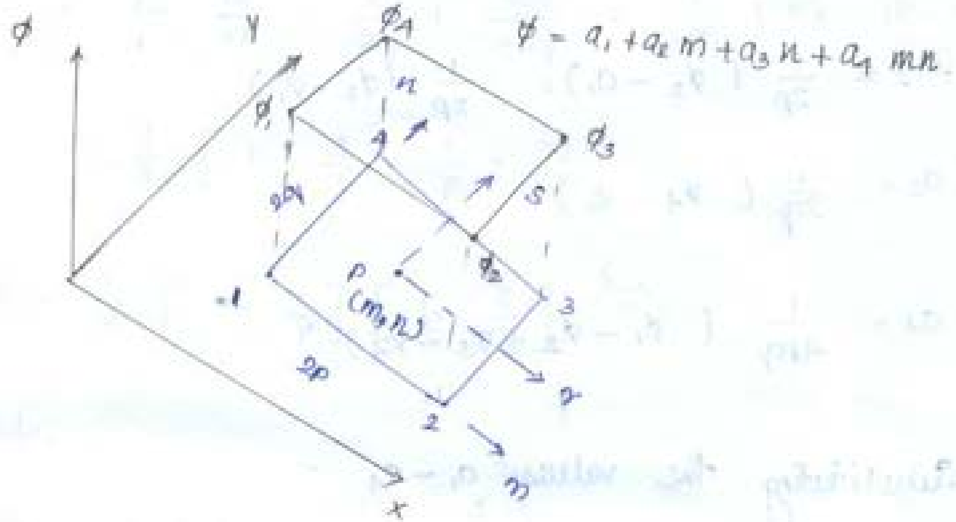
The contour for 46°C intersects the side 1-2 at the point (6.5, 3.5) and the side 2-3 at the point (5, 4.5).

In this contour line, temperature is constant (65°C) and hence called "Isotherm" line

Shape functions for Bilinear Rectangular element

(Two dimensional multiplex Element)

Bilinear rectangular element, whose boundaries are parallel to the coordinate axes.



$\phi = a_1 + a_2 m + a_3 n + a_4 mn$

let, m and n be the local coordinates whose origin is at node 1.

ϕ, field variable at any point P. In polynomial function.

$$\phi(m, n) = a_1 + a_2 m + a_3 n + a_4 mn.$$

$a_1 - a_4$ polynomial co-efficient.

No. of polynomial co-efficients = No. of degrees of freedom.

Nodal conditions,

$\phi = \phi_1$ at $m = n = 0$

$\phi = \phi_2$ at $m = 2p$; $n = 0$

$\phi = \phi_3$ at $m = 2p$; $n = 2q$

$\phi = \phi_4$ at $m = 0$; $n = 2q$

Substituting the nodal values,

$$\phi_1 = a_1$$

$$\phi_2 = a_1 + a_2 (2p)$$

$$\phi_3 = a_1 + a_2 (2p) + a_3 (2q) + a_4 (4pq)$$

$$\phi_4 = a_1 + a_3 (2q).$$

$$a_2 = \frac{1}{2p} (\phi_2 - a_1) = \frac{1}{2p} (\phi_2 - \phi_1)$$

$$a_3 = \frac{1}{2q} (\phi_4 - \phi_1)$$

$$a_4 = \frac{1}{4pq} (\phi_1 - \phi_2 + \phi_3 - \phi_4).$$

Substituting the values $a_1 - a_4$.

$$\phi(m,n) = \phi_1 + \frac{m}{2p} (\phi_2 - \phi_1) + \frac{n}{2q} (\phi_4 - \phi_1) +$$

$$\frac{mn}{4pq} (\phi_1 - \phi_2 + \phi_3 - \phi_4)$$

$$= \left(1 - \frac{m}{2p}\right)\left(1 - \frac{n}{2q}\right) \phi_1 +$$

$$\frac{m}{2p} \left(1 - \frac{n}{2q}\right) \phi_2 + \frac{mn}{4pq} \phi_3 + \frac{n}{2q} \left(1 - \frac{m}{2p}\right) \phi_4$$

$$= N_1 \phi_1 + N_2 \phi_2 + N_3 \phi_3 + N_4 \phi_4$$

$$\therefore \quad N_1 = \left(1 - \frac{m}{2p}\right)\left(1 - \frac{n}{2q}\right)$$

$$N_2 = \frac{m}{2p} \left(1 - \frac{n}{2q}\right)$$

$$N_3 = \frac{mn}{4pq}$$

$$N_4 = \frac{n}{2q} \left(1 - \frac{m}{2p}\right)$$

If the field variable ϕ is expressed interms of another et of local co-ordinates r and s, whose origin is at the centre.

let, $m = p + r$; $n = q + s$

$$N_1 = \left(1 - \frac{m}{2p}\right)\left(1 - \frac{n}{2q}\right)$$

$$= 1 - \frac{m}{2p} - \frac{n}{2q} + \frac{mn}{4pq} = \frac{1}{4pq}\left[pq - ps - qr + rs\right]$$

$$= \frac{1}{4}\left[1 - \frac{s}{q} - \frac{r}{p} + \frac{rs}{pq}\right]$$

$$N_1 = \frac{1}{4}\left[1 - \frac{r}{p}\right]\left[1 - \frac{s}{q}\right]$$

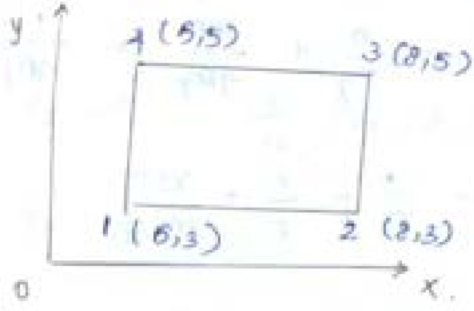
Similarly,

$$N_2 = \frac{1}{4}\left(1 + \frac{r}{p}\right)\left(1 - \frac{s}{q}\right)$$

$$N_3 = \frac{1}{4}\left(1 + \frac{r}{p}\right)\left(1 + \frac{s}{q}\right)$$

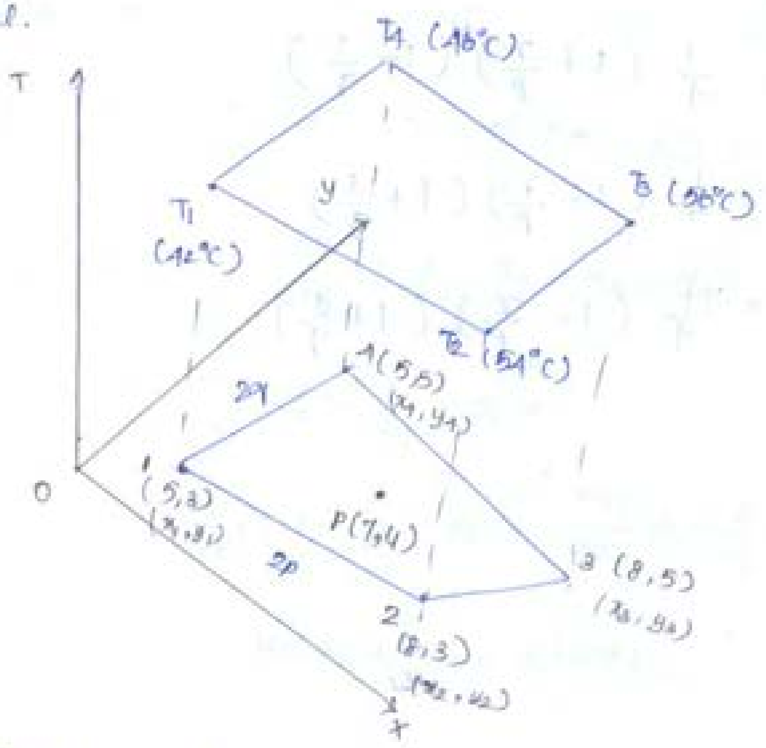
$$N_4 = \frac{1}{4}\left(1 - \frac{r}{p}\right)\left(1 + \frac{s}{q}\right).$$

For a four noded rectangular element shown in fig, determine the temperature at the point (7,4). The nodal values of temp. are $T_1 = 42°c$; $T_2 = 54°c$; $T_3 = 56°c$ and $T_4 = 46°c$; Also determine 3 points on the 50°c contour line.



Solution:-

The given diagram is redrawn to show the nodal temperature.



Let, m, n be the local co-ordinates for the given rectangular element and 2p, 2q be the length + width.

variation of temperature T on the element,

$$T = T_1 N_1 + T_2 N_2 + T_3 N_3 + T_4 N_4.$$

Shape functions are,

$$N_1 = \left(1 - \frac{m}{2p}\right)\left(1 - \frac{n}{2q}\right) ; \quad N_2 = \frac{m}{2p}\left(1 - \frac{n}{2q}\right)$$

$$N_3 = \frac{mn}{4pq} ; \quad N_4 = \frac{n}{2q}\left(1 - \frac{m}{2p}\right)$$

Now,

$$2p = x_2 - x_1 = 8 - 5 = 3$$

$$2q = y_2 - y_1 = 5 - 3 = 2$$

The point $(7, 4)$ in global co-ordinates (x, y) is changed in local coordinates (m, n) as,

$$m = x - x_1 = 7 - 5 = 2$$

$$n = y - y_1 = 4 - 3 = 1$$

Substituting in the shape functions,

$$N_1 = \left(1 - \frac{2}{3}\right)\left(1 - \frac{1}{2}\right) = \frac{1}{6} ;$$

$$N_2 = \frac{2}{3}\left(1 - \frac{1}{2}\right) = \frac{1}{3} ;$$

$$N_3 = \frac{1}{4} ; \quad N_4 = \frac{1}{2}\left(1 - \frac{2}{3}\right) = \frac{1}{6} .$$

The temperature at the point $(2, 1)$ is obtained using,

$$T = \left(\frac{1}{6} \times 42\right) + \left(\frac{1}{3} \times 54\right) + \left(\frac{1}{4} \times 56\right) + \left(\frac{1}{6} \times 46\right)$$

$$T = 51.4\,^\circ C$$

The contour line for $50^\circ C$:

Let, (x_a, y_a) ; (x_b, y_b) and (x_c, y_c) are the 3 required points in line 1-2, 3-4 4 (in between)

Along the line 1-2:

$$\frac{x_a - x_1}{x_2 - x_1} = \frac{T_a - T_1}{T_2 - T_1}$$

$$\frac{x_a - 5}{8-5} = \frac{50-42}{54-42} \quad ; \quad x_a = 7 \text{ cm}$$

and,

$$\frac{y_a - y_1}{y_2 - y_1} = \frac{T_a - T_1}{T_2 - T_1}$$

$$\frac{y_a - 3}{3-3} = \frac{50-42}{54-42} \quad ; \quad y_a = 3 \text{ cm};$$

Along the side 3-4:

$$\frac{x_c - x_4}{x_3 - x_4} = \frac{T_c - T_4}{T_3 - T_4}$$

$$\frac{x_c - 5}{8-5} = \frac{50-46}{56-46} \quad ; \quad x_c = 6.2 \text{ cm}$$

y_c = 5cm (line 3-4 is parallel to x-axis).

To find the third point, let $y_b = 4$ cm (centre line b/w 1-2 & 3-4)

$$n = y_b - y_1 = 4 - 3 = 1;$$

$$T = N_1 T_1 + N_2 T_2 + N_3 T_3 + N_4 T_4$$

$$50 = \left(1 - \frac{m}{2p}\right)\left(1 - \frac{n}{2q}\right) T_1 + \frac{m}{2p}\left(1 - \frac{n}{2q}\right) T_2 +$$

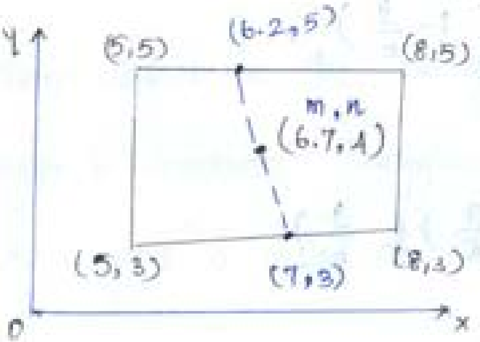
$$\frac{mn}{4pq} T_3 + \frac{n}{2q}\left(1 - \frac{m}{2p}\right)$$

$$50 = \left(1 - \frac{m}{3}\right)\left(1 - \frac{1}{2}\right) 42 + \frac{m}{3}\left(1 - \frac{1}{2}\right) 54 +$$

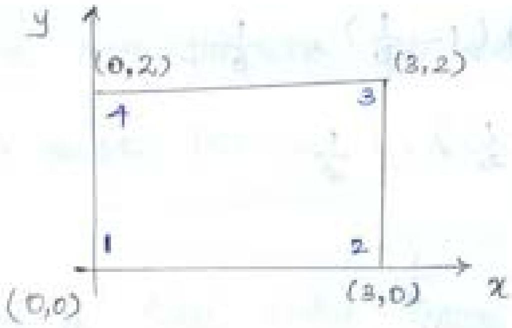
$$\left(\frac{m}{6}\right) 56 + \frac{1}{2}\left(1 - \frac{m}{3}\right) 46$$

$m = 1.7$ cm;

$x_b = x_1 + m = 5 + 1.7 = 6.7$ cm

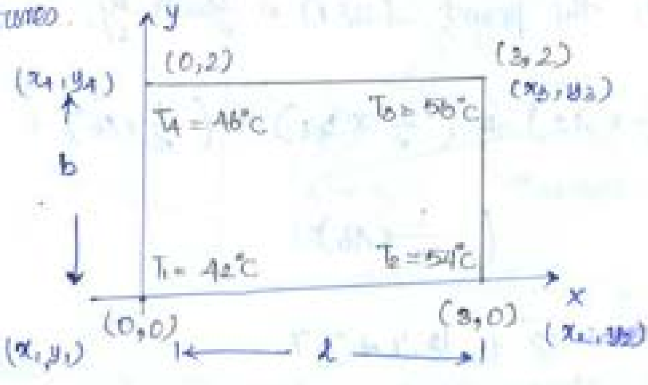


For a 4 noded rectangular element shown in fig. determine the temperature at the point (2,1). The nodal temperatures are $T_1 = 42°C$; $T_2 = 54°C$; $T_3 = 56°C$; $T_4 = 46°C$.



Solution:

The given diagram is redrawn to show the nodal temperatures.



Variation of temperature T on the element.

$$T = T_1 N_1 + T_2 N_2 + T_3 N_3 + T_4 N_4.$$

Shape functions are,

$$N_1 = \left(1 - \frac{x}{l}\right)\left(1 - \frac{y}{b}\right)$$

$$N_2 = \left(\frac{x}{l}\right)\left(1 - \frac{y}{b}\right)$$

$$N_3 = \frac{xy}{lb}$$

$$N_4 = \left(1 - \frac{x}{l}\right)\left(\frac{y}{b}\right)$$

From the figure,

$$L = x_2 - x_1 = 3 - 0 = 3$$

$$b = y_4 - y_1 = 2 - 0 = 2$$

Shape functions,

$$N_1 = \left(1 - \frac{2}{3}\right)\left(1 - \frac{1}{2}\right) = \frac{1}{6}$$

$$N_2 = \frac{2}{3}\left(1 - \frac{1}{2}\right) = \frac{1}{3}$$

$$N_3 = \frac{2 \times 1}{3 \times 2} = \frac{1}{3}$$

$$N_4 = \left(1 - \frac{2}{3}\right)\left(\frac{1}{2}\right) = \frac{1}{6};$$

Temperature at the point (2, 1) is given by,

$$T = \left(\frac{1}{6} \times 42\right) + \left(\frac{1}{3} \times 54\right) + \left(\frac{1}{3} \times 56\right) + \left(\frac{1}{6} \times 46\right)$$

$$= 7 + 18 + 18.7 + 7.7$$

$$T = 51.4°C.$$

CST and LST Elements

In LST, the displacement is assumed to vary linearly

The strain which is the change of displacement per unit length is constant throughout the element and hence this LST is called as CST.

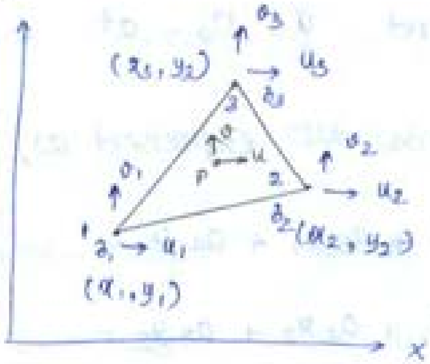
Application:

plates under bi-axial loadings and

bending of plates.

Derivation of shape functions for 2D linear Element related to vector problems (for constant strain triangular CST Element)

Let consider a three noded linear triangular CST element whose nodes may be specified as 1, 2 & 3 as shown in fig.



consider, δ_1, δ_2 & δ_3 → displacements at nodes 1, 2 & 3.

u_1, u_2, u_3 → components of displacements along x-axis

v_1, v_2, v_3 → ,, y-axis.

(x_1, y_1) (x_2, y_2) & (x_3, y_3) co-ordinates at 1, 2 & 3.

Nodal displacements,

$$\{\delta\} = \begin{Bmatrix} \delta_x \\ \delta_y \end{Bmatrix} = \begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ v_1 \\ v_2 \\ v_3 \end{Bmatrix}$$

For the linear element, the displacements u and v are linearly varying in the element and their values at any point (P) can be expressed as,

$$u(x, y) = a_1 + a_2 x + a_3 y \text{ —— } ①$$

$$v(x, y) = a_4 + a_5 x + a_6 y.$$

Since, CST element has got two degrees of freedom at each node, there are totally six degrees of freedom

now, applying the nodal conditions,

$u = u_1$ and $v = v_1$ at $x = x_1$ + $y = y_1$

$u = u_2$ and $v = v_2$, at $x = x_2$ & $y = y_2$

$u = u_3$ and $v = v_3$ at $x = x_3$ & $y = y_3$

Nodal displacements are expressed as,

$$u_1 = a_1 + a_2 x_1 + a_3 y_1$$

$$u_2 = a_1 + a_2 x_2 + a_3 y_2$$

$$u_3 = a_1 + a_3 x_3 + a_3 y_3$$ and

$$v_1 = a_4 + a_5 x_1 + a_6 y_1$$

$$v_2 = a_4 + a_5 x_2 + a_6 y_2$$

$$v_3 = a_4 + a_5 x_3 + a_6 y_3$$

In matrix form,

$$\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix}$$

$$\begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix}^{-1} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} \quad \text{(or)}$$

$$\{a\} = [D]^{-1} \{u\} \quad \text{— ②}$$

where,

$$[D] = \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix} = \text{co-ordinate matrix} \quad \text{and,}$$

$$[D]^{-1} = \frac{[C]^T}{|D|} \qquad C \to \text{co-efficient / co-factor matrix.}$$

$$[C] = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$$

$C_{11} = x_2 y_3 - x_3 y_2$; $C_{21} = x_3 y_1 - x_1 y_3$; $C_{31} = x_1 y_2 - x_2 y_1$

$C_{12} = y_2 - y_3$; $C_{22} = y_3 - y_1$; $C_{32} = y_1 - y_2$

$C_{13} = x_3 - x_2$; $C_{23} = x_1 - x_3$; $C_{33} = x_2 - x_1$

$$[C] = \begin{bmatrix} x_2 y_3 - x_3 y_2 & y_2 - y_3 & x_3 - x_2 \\ x_3 y_1 - x_1 y_3 & y_3 - y_1 & x_1 - x_3 \\ x_1 y_2 - x_2 y_1 & y_1 - y_2 & x_2 - x_1 \end{bmatrix}$$

$$[C]^T = \begin{bmatrix} x_2 y_3 - x_3 y_2 & x_3 y_1 - x_1 y_3 & x_1 y_2 - x_2 y_1 \\ y_2 - y_3 & y_3 - y_1 & y_1 - y_2 \\ x_3 - x_2 & x_1 - x_3 & x_2 - x_1 \end{bmatrix}$$

determinant of matrix, D

$$|D| = \begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix} = (x_2 y_3 - x_3 y_2) - x_1 (y_3 - y_2) + y_1 (x_3 - x_2)$$

$$|D| = 2A$$

$\therefore$ Equ (2) $\rightarrow$ $\{a\} = [D]^{-1}\{u\} = \dfrac{[C]^T}{|D|}\{u\}$

$$\begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} = \frac{1}{2A}\begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

where,

$\alpha_1 = x_2 y_3 - x_3 y_2$; $\alpha_2 = x_3 y_1 - x_1 y_3$; $\alpha_3 = x_1 y_2 - x_2 y_1$

$\beta_1 = y_2 - y_3$; $\beta_2 = y_3 - y_1$; $\beta_3 = y_1 - y_2$

$\gamma_1 = x_3 - x_2$; $\gamma_2 = x_1 - x_3$; $\gamma_3 = x_2 - x_1$

Equ (1) $\rightarrow$ In matrix form,

$$u(x, y) = \begin{bmatrix} 1 & x & y \end{bmatrix}\begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix}$$

$$= \begin{bmatrix} 1 & x & y \end{bmatrix}\frac{1}{2A}\begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

$$= \frac{1}{2A}\begin{bmatrix} \alpha_1 + \beta_1 x + \gamma_1 y & \alpha_2 + \beta_2 x + \gamma_2 y & \alpha_3 + \beta_3 x + \gamma_3 y \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

$$u = N_1 u_1 + N_2 u_2 + N_3 u_3 \quad \text{— (3)}$$

N_1, N_2 & N_3 are shape functions,

$$N_1 = \frac{\alpha_1 + \beta_1 x + \gamma_1 y}{2A} ; \qquad N_2 = \frac{\alpha_2 + \beta_2 x + \gamma_2 y}{2A}$$

$$N_3 = \frac{\alpha_3 + \beta_3 x + \gamma_3 y}{2A}$$

The same procedure is followed for the displacement v at the nodes 1, 2, 3 we get,

$$v = N_1 v_1 + N_2 v_2 + N_3 v_3 . \quad — ④$$

N_1, N_2, N_3 are the shape functions and they have the same values are obtained.

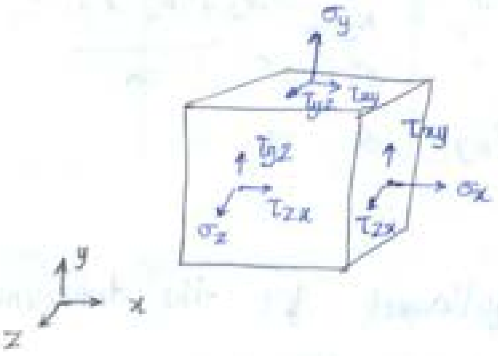
combining eqn (3) & (4),

$$\text{Displacement } \delta(x, y)_p = \begin{Bmatrix} u(x, y) \\ v(x, y) \end{Bmatrix}_p$$

$$= \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix}$$

$$\delta_p = [N] \{\delta\}$$

Stress-strain Relations (Elasticity equations).



let us consider a 3 Dimensional body, Subjected to

normal stresses σ_x, σ_y and σ_z

Shear stress τ_{xy}, τ_{yz} and τ_{zx}

Stresses acting on the body,

$$\{\sigma\} = \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{Bmatrix} \longleftarrow (1)$$

Strains,

$$\{e\} = \begin{Bmatrix} e_x \\ e_y \\ e_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix}$$

e_x, e_y & e_z → Normal strains

γ_{xy}, γ_{yz}, γ_{zx} → shear strains.

stress - strain Relationship matrix - constitutive matrix.

Now consider normal stress conditions.

Net strain occured in x-direction;

$$e_x = \frac{\sigma_x}{E} - \mu \frac{\sigma_y}{E} - \mu \frac{\sigma_z}{E} ;$$

$\mu \rightarrow$ Poission ratio.

similarly,

$$e_y = -\mu \frac{\sigma_x}{E} + \frac{\sigma_y}{E} - \mu \cdot \frac{\sigma_z}{E}$$

$$e_z = -\mu \frac{\sigma_x}{E} - \mu \cdot \frac{\sigma_y}{E} + \frac{\sigma_z}{E}$$

To solve σ_x, σ_y and σ_z: and got the solution:

$$\sigma_x = \frac{E}{(1+\mu)(1-2\mu)} \left[e_x (1-\mu) + \mu \cdot e_y + \mu e_z \right]$$

$$\sigma_y = \frac{E}{(1+\mu)(1-2\mu)} \left[\mu \cdot e_x + (1-\mu) e_y + \mu e_z \right]$$

$$e_z = \frac{E}{(1+\mu)(1-2\mu)} \left[\mu \cdot e_x + \mu e_y + (1-\mu) e_z \right]$$

shear stress - shear strain relations:

shear stress, $\tau = G \cdot \gamma$ $G \rightarrow$ Modulus of rigidity.

$$\tau_{xy} = G \cdot \gamma_{xy} ; \quad \tau_{yz} = G \cdot \gamma_{yz} ; \quad \tau_{xz} = G \cdot \gamma_{zx} .$$

Replacing G by E; $G = \frac{E}{2(1+\mu)}$

$$\tau_{xy} = \frac{E}{2(1+\mu)} \gamma_{xy}$$

$$= \frac{E}{(1+\mu)(1-2\mu)} \left[\frac{1-2\mu}{2} \right] \gamma_{xy}$$

Similarly,

$$\tau_{yz} = \frac{E}{(1+\mu)(1-2\mu)} \left[\frac{1-2\mu}{2} \right] \gamma_{yz}$$

$$\tau_{zx} = \frac{E}{(1+\mu)(1-2\mu)} \left[\frac{1-2\mu}{2} \right] \gamma_{zx} .$$

The above equations are called as "Elasticity Equations".

Applying in the Eq (1)

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{Bmatrix} = \frac{E}{(1+\mu)(1-2\mu)} \begin{bmatrix} 1-\mu & \mu & \mu & 0 & 0 & 0 \\ \mu & 1-\mu & \mu & 0 & 0 & 0 \\ \mu & \mu & 1-\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1-2\mu}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1-2\mu}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1-2\mu}{\mu} \end{bmatrix} \begin{Bmatrix} e_x \\ e_y \\ e_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{xz} \end{Bmatrix}$$

The above equation can be expressed as,

$$\{\sigma\} = [D] \{e\} .$$

[D] → stress - strain / Elasticity / constitutive Matrix.

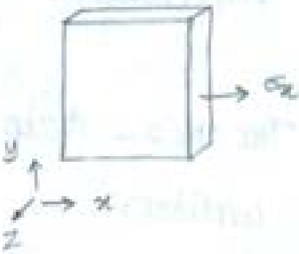
$$[D] = \frac{E}{(1+\mu)(1-2\mu)} \begin{bmatrix} 1-\mu & \mu & \mu & & & \\ \mu & 1-\mu & \mu & 0 & 0 & 0 \\ \mu & \mu & 1-\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.5-\mu & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.5-\mu & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.5-\mu \end{bmatrix}_{(6\times6)}$$

Plane stress:-

A state of plane stress is said to exist when the elastic body is very thin and there are no loads applied in the co-ordinate direction parallel to the thickness.

Ex: Plates with holes, fillet or other changes in geometry

For plane-stressed element



$$e_x = \frac{\sigma_x}{E} - \mu \frac{\sigma_y}{E}$$

$$e_y = -\mu \cdot \frac{\sigma_x}{E} + \frac{\sigma_y}{E}$$

$$e_z = -\mu \cdot \frac{\sigma_x}{E} - \mu \frac{\sigma_y}{E}$$

$$\gamma_{xy} = \frac{2(1+\mu)}{E} \cdot \tau_{xy}$$

$$\sigma_x = \frac{E}{(1-\mu^2)} (e_x + \mu \cdot e_y)$$

$$\sigma_y = \frac{E}{(1-\mu^2)} (\mu \cdot e_x + e_y)$$

$$\tau_{xy} = \frac{E}{2(1+\mu)} \gamma_{xy} = \frac{E}{(1-\mu^2)} \left(\frac{1-\mu}{2}\right) \gamma_{xy}$$

The equations in matrix form,

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{E}{(1-\mu^2)} \begin{bmatrix} 1 & \mu & 0 \\ \mu & 1 & 0 \\ 0 & 0 & \frac{1-\mu}{2} \end{bmatrix} \begin{Bmatrix} e_x \\ e_y \\ \gamma_{xy} \end{Bmatrix}$$

$$\{\sigma\} = [D]\{e\}$$

$$[D] = \frac{E}{(1-\mu^2)} \begin{bmatrix} 1 & \mu & 0 \\ \mu & 1 & 0 \\ 0 & 0 & \frac{1-\mu}{2} \end{bmatrix}$$

∴ the constitutive Matrix for plane stress condition.

Plane strain:

The state of plane strain occurs in members that are not free to expand in the direction perpendicular to the plane of applied loads.

The stress-strain relationship matrix, for plane strain condition:

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{xz} \end{Bmatrix} = \frac{E}{(1+\mu)(1-2\mu)} \begin{bmatrix} 1-\mu & \mu & \mu & 0 & 0 & 0 \\ \mu & 1-\mu & \mu & 0 & 0 & 0 \\ \mu & \mu & 1-\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1-2\mu}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1-2\mu}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1-2\mu}{2} \end{bmatrix} \begin{Bmatrix} e_x \\ e_y \\ 0 \\ \gamma_{xy} \\ 0 \\ 0 \end{Bmatrix}$$

← ①

Since, $e_z = \gamma_{yz} = \gamma_{zx} = 0$, neglecting 3rd, 5th & 6th rows,

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{E}{(1+\mu)(1-2\mu)} \begin{bmatrix} 1-\mu & \mu & 0 \\ \mu & 1-\mu & 0 \\ 0 & 0 & \frac{1-2\mu}{\mu} \end{bmatrix} \begin{Bmatrix} e_x \\ e_y \\ \gamma_{xy} \end{Bmatrix}$$

$$\{\sigma\} = [D]\{e\}$$

$$[D] = \frac{E}{(1+\mu)(1-2\mu)} \begin{bmatrix} 1-\mu & \mu & 0 \\ \mu & 1-\mu & 0 \\ 0 & 0 & \frac{1-2\mu}{2} \end{bmatrix}$$

from Eqn (1).

$$\sigma_z = \frac{E}{(1+\mu)(1-2\mu)} (\mu e_x + \mu e_y).$$

$$\sigma_z = \frac{E\mu(e_x + e_y)}{(1+\mu)(1-2\mu)}$$

Strain - displacement relationship matrix.

Shape functions are,

$$N_1 = \frac{1}{2A}(\alpha_1 + \beta_1 x + \gamma_1 y)$$

$$N_2 = \frac{1}{2A}(\alpha_2 + \beta_2 x + \gamma_2 y)$$

$$N_3 = \frac{1}{2A}(\alpha_3 + \beta_3 x + \gamma_3 y)$$

Strain,

In x - direction, $e_x = \frac{\partial u}{\partial x} = \frac{\partial N_1}{\partial x} u_1 + \frac{\partial N_2}{\partial x} u_2 + \frac{\partial N_3}{\partial x} u_3$

where, $u = u_1 N_1 + u_2 N_2 + u_3 N_3$.

$$e_y = \frac{\partial v}{\partial y} = \frac{\partial N_1}{\partial y} v_1 + \frac{\partial N_2}{\partial y} v_2 + \frac{\partial N_3}{\partial y} v_3$$

Shear-strain, $\gamma_{xy} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial x}$

$$= \frac{\partial N_1}{\partial y} u_1 + \frac{\partial N_2}{\partial y} u_2 + \frac{\partial N_3}{\partial y} u_3 + \frac{\partial N_1}{\partial x} v_1 + \frac{\partial N_2}{\partial x} v_2 + \frac{\partial N_3}{\partial x} v_3$$

In matrix form,

$$\begin{Bmatrix} e_x \\ e_y \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} \frac{\partial N_1}{\partial x} & 0 & \frac{\partial N_2}{\partial x} & 0 & \frac{\partial N_3}{\partial x} & 0 \\ 0 & \frac{\partial N_1}{\partial y} & 0 & \frac{\partial N_2}{\partial y} & 0 & \frac{\partial N_3}{\partial y} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial y} & \frac{\partial N_2}{\partial x} & \frac{\partial N_3}{\partial y} & \frac{\partial N_3}{\partial x} \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix}$$

from the shape function,

$$\frac{\partial N_1}{\partial x} = \frac{\beta_1}{2A} ; \quad \frac{\partial N_2}{\partial x} = \frac{\beta_2}{2A} ; \quad \frac{\partial N_3}{\partial x} = \frac{\beta_3}{2A}$$

$$\frac{\partial N_1}{\partial y} = \frac{\gamma_1}{2A} ; \quad \frac{\partial N_2}{\partial y} = \frac{\gamma_2}{2A} ; \quad \frac{\partial N_3}{\partial y} = \frac{\gamma_3}{2A}$$

$$\begin{Bmatrix} e_x \\ e_y \\ \gamma_{xy} \end{Bmatrix} = \frac{1}{2A} \begin{bmatrix} \beta_1 & 0 & \beta_2 & 0 & \beta_3 & 0 \\ 0 & \gamma_1 & 0 & \gamma_2 & 0 & \gamma_3 \\ \gamma_1 & \beta_1 & \gamma_2 & \beta_2 & \gamma_3 & \beta_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix}$$

$$\{e\} = [B] [\delta]$$

strain-displacement matrix, $[B] = \frac{1}{2A} \begin{bmatrix} \beta_1 & 0 & \beta_2 & 0 & \beta_3 & 0 \\ 0 & \gamma_1 & 0 & \gamma_2 & 0 & \gamma_3 \\ \gamma_1 & \beta_1 & \gamma_2 & \beta_2 & \gamma_3 & \beta_3 \end{bmatrix}$

where, $\beta_1 = y_2 - y_3$; $\beta_2 = y_3 - y_1$; $\beta_3 = y_1 - y_2$

$\gamma_1 = x_3 - x_2$; $\gamma_2 = x_1 - x_3$; $\gamma_3 = x_2 - x_1$

Stress-displacement Relations:

stress, $\{\sigma\} = [D]\{e\}$; $\{e\} = [B]\{\delta\}$

$$\therefore \{\sigma\} = [D][B]\{\delta\}$$

For plane stress condition,

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{E}{(1-\mu^2)} \begin{bmatrix} 1 & \mu & 0 \\ \mu & 1 & 0 \\ 0 & 0 & \frac{1-\mu}{2} \end{bmatrix} \frac{1}{2A} \begin{bmatrix} \beta_1 & 0 & \beta_2 & 0 & \beta_3 & 0 \\ 0 & \gamma_1 & 0 & \gamma_2 & 0 & \gamma_3 \\ \gamma_1 & \beta_1 & \gamma_2 & \beta_2 & \gamma_3 & \beta_3 \end{bmatrix} \times \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix}$$

For plane strain condition,

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{E}{(1+\mu)(1-2\mu)} \begin{bmatrix} 1-\mu & \mu & 0 \\ \mu & 1-\mu & 0 \\ 0 & 0 & \frac{1-2\mu}{2} \end{bmatrix} \times$$

$$\frac{1}{2A} \begin{bmatrix} \beta_1 & 0 & \beta_2 & 0 & \beta_3 & 0 \\ 0 & \gamma_1 & 0 & \gamma_2 & 0 & \gamma_3 \\ \gamma_1 & \beta_1 & \gamma_2 & \beta_2 & \gamma_3 & \beta_3 \end{bmatrix} \times \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix}$$

The maximum & minimum normal stress & maximum shear stress.

1. Maximum normal stress,

$$\sigma_1 = \frac{1}{2}\left[(\sigma_x+\sigma_y)+\sqrt{(\sigma_x-\sigma_y)^2+4\tau_{xy}^2}\right]$$

2. Minimum normal stress,

$$\sigma_2 = \frac{1}{2}\left[(\sigma_x+\sigma_y)-\sqrt{(\sigma_x-\sigma_y)^2+4\tau_{xy}^2}\right.$$

3. Maximum shear stress,

$$\tau_m = \frac{1}{2}(\sigma_1-\sigma_2) = \frac{1}{2}\sqrt{(\sigma_x-\sigma_y)^2+4\tau_{xy}^2}$$

4. Principal Angle θ_p:

$$\tan 2\theta_p = \frac{2\tau_{xy}}{(\sigma_x-\sigma_y)}$$

$$\theta_p = \frac{1}{2}\tan^{-1}\left[\frac{2\tau_{xy}}{\sigma_x-\sigma_y}\right]$$

Stiffness matrix for 2D (CST) Element.

stiffness matrix, $[K] = \int_v [B]^T[D][B].dv$

$$[K] = [B]^T[D][B].A.t$$

Temperature effects on CST Elements.

In a CST Element, if the temperature is higher or lower temperature than room temperature, the change in temperature ΔT produces some amount of deformation is known as "thermal strain".

Initial strain (or) thermal strain

$$\{e_0\} = \begin{Bmatrix} e_{x0} \\ e_{y0} \\ \gamma_{xy0} \end{Bmatrix} = \begin{Bmatrix} \alpha \Delta T \\ \alpha \Delta T \\ 0 \end{Bmatrix}$$

(Plane stress condition)

For plane strain condition,

$$\{e_0\} = \begin{Bmatrix} e_{x0} \\ e_{y0} \\ \gamma_{xy0} \end{Bmatrix} = (1+\mu) \begin{Bmatrix} \alpha \Delta T \\ \alpha \Delta T \\ 0 \end{Bmatrix}$$

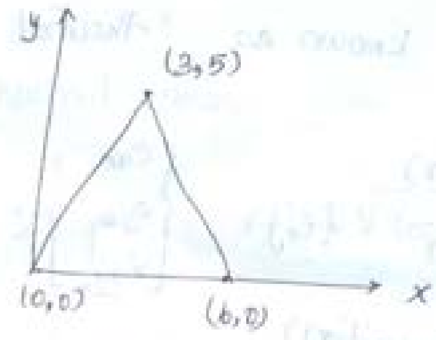
Thermal load, $\{F_0\}$

$$\{F_0\} = [B]^T [D] \{e_0\} At.$$

(6×3) (3×3) (3×1) → (6×1) matrix.

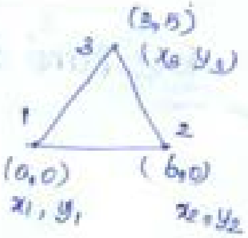
Evaluate the element stiffness matrix for the triangular element shown in fig under plane strain condition. Assume the following values

$E = 200\ GPa;\ \mu = 0.25;\ t = 1mm;$



Solution:-

The nodal co-ordinates are,



$x_1 = 0;\ y_1 = 0;$

$x_2 = 6;\ y_2 = 0$

$x_3 = 3;\ y_3 = 5.$

$E = 200\ GPa = 2 \times 10^5\ N/mm^2;\ t = 1\ mm;\ \mu = 0.25.$

Element stiffness matrix, $[K] = [B]^T [D] [B]\ A \cdot t.$

$$\text{Area, } A = \frac{1}{2} \begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}$$

$$= \frac{1}{2} \begin{vmatrix} 1 & 0 & 0 \\ 1 & 6 & 0 \\ 1 & 3 & 5 \end{vmatrix}$$

$$= \frac{1}{2} \left[1(30-0) - 0 + 0 \right] = \frac{30}{2} = 15 mm^2$$

Strain-displacement matrix [B]:

$$[B] = \frac{1}{2A}\begin{bmatrix} \beta_1 & 0 & \beta_2 & 0 & \beta_3 & 0 \\ 0 & \gamma_1 & 0 & \gamma_2 & 0 & \gamma_3 \\ \gamma_1 & \beta_1 & \gamma_2 & \beta_2 & \gamma_3 & \beta_3 \end{bmatrix}$$

$\beta_1 = y_2 - y_3 = 0 - 5 = 0 - 5 = -5$; $\gamma_1 = x_3 - x_2 = 3 - 6 = -3$

$\beta_2 = y_3 - y_1 = 5 - 0 = 5$; $\gamma_2 = x_1 - x_3 = 0 - 3 = -3$

$\beta_3 = y_1 - y_2 = 0$; $\gamma_3 = x_2 - x_1 = 6 - 0 = 6.$

$$[B] = \frac{1}{30}\begin{bmatrix} -5 & 0 & 5 & 0 & 0 & 0 \\ 0 & -3 & 0 & -3 & 0 & 6 \\ -3 & -5 & -3 & 5 & 6 & 0 \end{bmatrix}$$

$$[B]^T = \frac{1}{30}\begin{bmatrix} -5 & 0 & -3 \\ 0 & -3 & -5 \\ 5 & 0 & -3 \\ 0 & -3 & 5 \\ 0 & 0 & 6 \\ 0 & 6 & 0 \end{bmatrix}$$

For plane strain condition: [D] → stress-strain relationship matrix.

$$[D] = \frac{E}{(1+\mu)(1-2\mu)}\begin{bmatrix} 1-\mu & \mu & 0 \\ \mu & 1-\mu & 0 \\ 0 & 0 & \frac{1-2\mu}{\mu} \end{bmatrix}$$

$$= \frac{2\times10^5}{(1+0.25)(1-0.5)}\begin{bmatrix} 0.75 & 0.25 & 0 \\ 0.25 & 0.75 & 0 \\ 0 & 0 & 0.25 \end{bmatrix}$$

$$= 8\times10^4\begin{bmatrix} 3 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Stiffness matrix $[K] = [B]^T [D] [B] At$

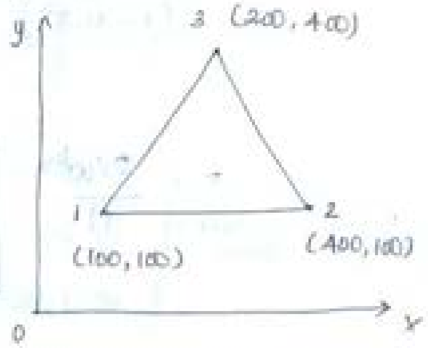
$$[K] = \frac{1}{30} \begin{bmatrix} -5 & 0 & -3 \\ 0 & -3 & -5 \\ 5 & 0 & -3 \\ 0 & -3 & 5 \\ 0 & 0 & 6 \\ 0 & 6 & 0 \end{bmatrix} 8\times10^4 \begin{bmatrix} 3 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \times$$

$$\frac{1}{30} \begin{bmatrix} -5 & 0 & +5 & 0 & 0 & 0 \\ 0 & -3 & 0 & -3 & 0 & 6 \\ -3 & -5 & -3 & 5 & 6 & 0 \end{bmatrix} \times (15\times1)$$

$$= \frac{8\times10^4\times15}{30\times30} \begin{bmatrix} -15 & -5 & -3 \\ -3 & -9 & -5 \\ 15 & 5 & -3 \\ -3 & -9 & 5 \\ 0 & 0 & 0 \\ 6 & 18 & 0 \end{bmatrix} \begin{bmatrix} -5 & 0 & 5 & 0 & 0 & 0 \\ 0 & -3 & 0 & -3 & 0 & 6 \\ -3 & -5 & -3 & 5 & 6 & 0 \end{bmatrix}$$

$$[K] = 1.33\times10^3 \begin{bmatrix} 84 & 30 & -66 & 0 & -18 & -30 \\ 30 & 52 & 0 & 2 & -30 & -54 \\ -66 & 0 & 84 & -30 & -18 & 30 \\ 0 & 2 & -30 & 52 & 30 & -54 \\ -18 & -30 & -18 & 30 & 36 & 0 \\ -30 & -54 & 30 & -54 & 0 & 108 \end{bmatrix} N/mm.$$

For the plane stress element shown in fig. the nodal displacements are,



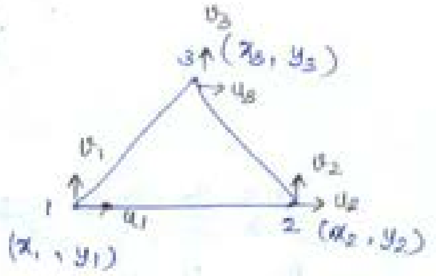
$u_1 = 2.0$ mm ; $v_1 = 1.0$ mm

$u_2 = 1.0$ mm ; $v_2 = 1.5$ mm

$u_3 = 2.5$ mm ; $v_3 = 0.5$ mm

Get the element stresses. Assume $E = 200$ GN/m² ; $\mu = 0.3$. $t = 10$ mm. All the co-ordinates are in mm.

Solution:-



$x_1 = 100$ mm; $y_1 = 100$ mm

$x_2 = 400$ mm; $y_2 = 100$ mm

$x_3 = 200$ mm; $y_3 = 400$ mm;

Element stress, $\{\sigma\} = [D]\{e\} = [D][B]\{\delta\}$

$$\{\sigma\} = \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}$$

Stress-strain relationship matrix, $[D] = \dfrac{E}{(1-\mu^2)} \begin{bmatrix} 1 & \mu & 0 \\ \mu & 1 & 0 \\ 0 & 0 & \frac{1-\mu}{2} \end{bmatrix}$

Plane stress condition.

Strain-displacement matrix

$$[B] = \frac{1}{2A} \begin{bmatrix} \beta_1 & 0 & \beta_2 & 0 & \beta_3 & 0 \\ 0 & \gamma_1 & 0 & \gamma_2 & 0 & \gamma_3 \\ \gamma_1 & \beta_1 & \gamma_2 & \beta_2 & \gamma_3 & \beta_3 \end{bmatrix}$$

Nodal-displacement vector:

$$\{\delta\}_n = \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix}$$

$$[D] = \frac{2\times10^5}{(1-0.3^2)}\begin{bmatrix}1 & 0.3 & 0\\ 0.3 & 1 & 0\\ 0 & 0 & \frac{1-0.3}{2}\end{bmatrix}$$

$$= \frac{2\times10^6}{91}\begin{bmatrix}10 & 3 & 0\\ 3 & 10 & 0\\ 0 & 0 & 3.5\end{bmatrix}$$

$$[B] \Rightarrow \quad A = \frac{1}{2}\begin{vmatrix}1 & x_1 & y_1\\ 1 & x_2 & y_2\\ 1 & x_3 & y_3\end{vmatrix}$$

$$= \frac{1}{2}\begin{vmatrix}1 & 200 & 100\\ 1 & 400 & 100\\ 1 & 200 & 400\end{vmatrix} = 45000\,mm^2$$

$\beta_1 = y_2 - y_3 = -300; \quad \gamma_1 = x_3 - x_2 = -200$

$\beta_2 = y_3 - y_1 = 300; \quad \gamma_2 = x_1 - x_3 = -100$

$\beta_3 = y_1 - y_2 = 0; \quad \gamma_3 = x_2 - x_1 = 200;$

$$\therefore [B] = \frac{1}{2\times 45000}\begin{bmatrix}-300 & 0 & 300 & 0 & 0 & 0\\ 0 & -200 & 0 & -100 & 0 & 300\\ -200 & -300 & -100 & 300 & 300 & 0\end{bmatrix}$$

$$= \frac{1}{900}\begin{bmatrix}-3 & 0 & 3 & 0 & 0 & 0\\ 0 & -2 & 0 & -1 & 0 & 3\\ -2 & -3 & -1 & 3 & 3 & 0\end{bmatrix}$$

$$\begin{Bmatrix}\sigma_x\\ \sigma_y\\ \tau_{xy}\end{Bmatrix} = \frac{2\times10^6}{91}\begin{bmatrix}10 & 3 & 0\\ 3 & 10 & 0\\ 0 & 0 & 3.5\end{bmatrix}\frac{1}{900}\begin{bmatrix}-3 & 0 & 3 & 0 & 0 & 0\\ 0 & -2 & 0 & -1 & 0 & 3\\ -2 & -3 & -1 & 3 & 3 & 0\end{bmatrix} \times \begin{Bmatrix}2.0\\ 1.0\\ 1.0\\ 1.5\\ 2.5\\ 0.5\end{Bmatrix}$$

$$= \frac{2\times10^6}{91\times900}\begin{bmatrix}10 & 3 & 0\\ 3 & 10 & 0\\ 0 & 0 & 3.5\end{bmatrix}\begin{Bmatrix}-3\\ -2\\ 4\end{Bmatrix}$$

$$= \frac{2\times10^4}{819}\begin{Bmatrix}-36\\ -29\\ 14\end{Bmatrix} = \begin{Bmatrix}-879.12\\ -708.18\\ 341.88\end{Bmatrix} N/mm^2$$

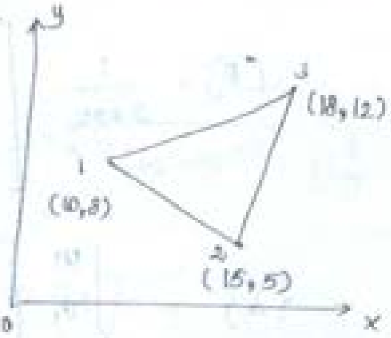
Element stresses are,

$\sigma_x = -879.12$ N/mm^2

$\sigma_y = -708.18$ N/mm^2

$\tau_{xy} = 341.88$ N/mm^2.

Calculate the element stresses σ_x, σ_y, τ_{xy}, σ_1 & σ_2 and the Principal angle θ_p for the CST element shown in fig. the nodal displacements are,



$u_1 = 2.0$ μm; $v_1 = 1.0$ μm

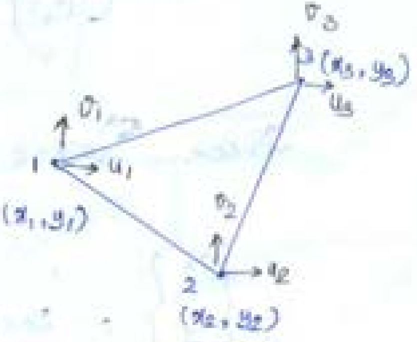
$u_2 = 0.5$ μm; $v_2 = 1.5$ μm

$u_3 = 1.2$ μm; $v_3 = 2.8$ μm

Take, E = 210 GPa; $\mu = 0.25$ Assume plane stress condition.

Solution:-

The element stress,



$$\{\sigma\} = [D][B]\{\delta\}$$

$$[D] = \frac{E}{(1-\mu^2)}\begin{bmatrix}1 & \mu & 0\\ \mu & 1 & 0\\ 0 & 0 & \frac{1-\mu}{2}\end{bmatrix}$$ for plane-stress condition.

$$= \frac{2.1\times10^5}{(1-0.25^2)}\begin{bmatrix} 1 & 0.25 & 0 \\ 0.25 & 1 & 0 \\ 0 & 0 & \frac{1-0.25}{2} \end{bmatrix}$$

$$= 56\times10^3\begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 1.5 \end{bmatrix}$$

$$[B] \Rightarrow \quad A = \begin{vmatrix} 1 & 10 & 8 \\ 1 & 15 & 5 \\ 1 & 18 & 12 \end{vmatrix} = 22\,mm^2$$

$\beta_1 = y_2 - y_3 = 5 - 12 = -7$; $\gamma_1 = x_3 - x_2 = 3$

$\beta_2 = y_3 - y_1 = 4$; $\gamma_2 = x_1 - x_3 = -8$

$\beta_3 = y_1 - y_2 = 3$; $\gamma_3 = x_2 - x_1 = 5$

$$[B] = \frac{1}{2\times22}\begin{bmatrix} -7 & 0 & 4 & 0 & 3 & 0 \\ 0 & 3 & 0 & -8 & 0 & 5 \\ 3 & -7 & -8 & 4 & 5 & 3 \end{bmatrix}$$

$$\{\delta\} = \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix} = \frac{1}{10^3}\begin{Bmatrix} 2 \\ 1 \\ 0.5 \\ 1.5 \\ 1.2 \\ 2.8 \end{Bmatrix}$$

Stresses, $\{\sigma\} = [D][B]\{\delta\}$

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = 56\times10^3\begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 1.5 \end{bmatrix}\frac{1}{44}\begin{bmatrix} -7 & 0 & 4 & 0 & 3 & 0 \\ 0 & 3 & 0 & -8 & 0 & 5 \\ 3 & -7 & -8 & 4 & 5 & 3 \end{bmatrix}\times\frac{1}{10^3}\begin{Bmatrix} 2 \\ 1 \\ 0.5 \\ 1.5 \\ 1.2 \\ 2.8 \end{Bmatrix}$$

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{56 \times 10^3}{44 \times 10^3} \begin{bmatrix} -28 & 3 & 16 & -8 & 12 & 5 \\ -7 & 12 & 4 & -32 & 3 & 20 \\ 4.5 & -10.5 & -12 & 6 & 7.5 & 4.5 \end{bmatrix} \begin{Bmatrix} 2 \\ 1 \\ 0.5 \\ 1.5 \\ 1.2 \\ 2.8 \end{Bmatrix}$$

$$= \frac{14}{11} \begin{bmatrix} -56 + 3 + 8 - 12 + 14.4 + 14 \\ -14 + 12 + 2 - 48 + 3.6 + 56 \\ 9 - 10.5 - 6 + 9 + 9 + 12.6 \end{bmatrix}$$

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \begin{Bmatrix} -36.4 \\ 14.76 \\ 29.4 \end{Bmatrix} N/mm^2$$

Maximum normal stress, $\sigma_1 = \frac{1}{2}\left[(\sigma_x + \sigma_y) + \sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}\right]$

$$\sigma_1 = \frac{1}{2}\left[(-21.64) + (77.94)\right]$$

$$= 28.15 \ N/mm^2$$

Minimum normal stress, $\sigma_2 = \frac{1}{2}\left[(\sigma_x + \sigma_y) + \sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}\right]$

$$= \frac{1}{2}\left[-21.64 - 77.94\right]$$

$$= -49.79 \ N/mm^2$$

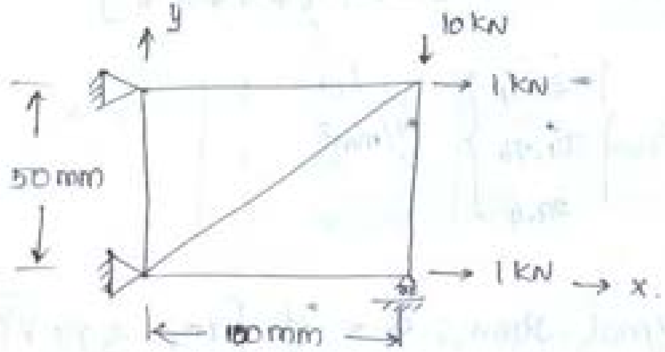
Principal angle $\theta_p = \frac{1}{2}\tan^{-1}\left[\frac{2\tau_{xy}}{\sigma_x - \sigma_y}\right]$

$$= \frac{1}{2}\tan^{-1}\left[\frac{2 \times 29.4}{-36.4 - 14.76}\right]$$

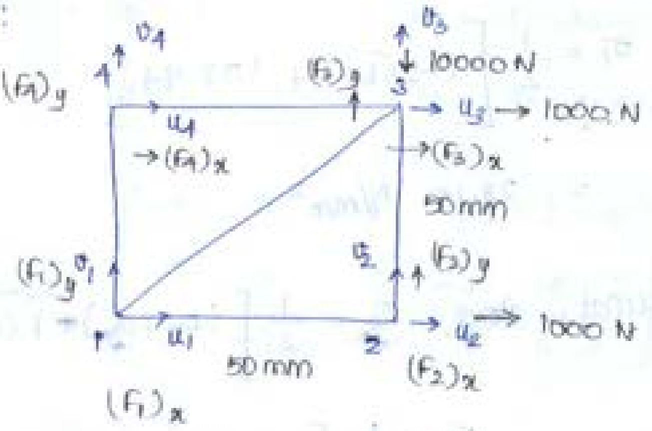
$$= -24.5^\circ$$

Det the nodal displacements + stresses for the two dimensional loaded plate as shown in fig. Assume the plane stress condition. Body force may be neglected in compairison to the external forces;

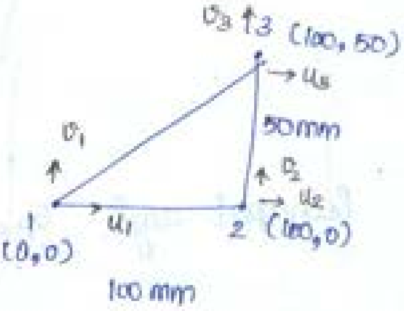
Take, $E = 210\ GPa$; $\mu = 0.25$; $t = 10\ mm$.



Solution:



Consider - Element ①



$x_1 = 0$; $y_1 = 0$;

$x_2 = 100mm$; $y_2 = 0$;

$x_3 = 100mm$; $y_3 = 50\ mm$;

$$[K]_1 = \left[[B]^T [D] [B] A t\right]_1$$

$$[B] = \frac{1}{2A}\begin{bmatrix} \beta_1 & 0 & \beta_2 & 0 & \beta_3 & 0 \\ 0 & \gamma_1 & 0 & \gamma_2 & 0 & \gamma_3 \\ \gamma_1 & \beta_1 & \gamma_2 & \beta_2 & \gamma_3 & \beta_3 \end{bmatrix}$$

Area, $A = \begin{vmatrix} 1 & 0 & 0 \\ 1 & 100 & 0 \\ 1 & 100 & 50 \end{vmatrix} = 2500 \text{ mm}^2$

$\beta_1 = y_2 - y_3 = -50$; $\gamma_1 = x_3 - x_2 = 0$

$\beta_2 = y_3 - y_1 = 50$; $\gamma_2 = x_1 - x_3 = -100$

$\beta_3 = y_1 - y_2 = 0$; $\gamma_3 = x_2 - x_1 = 100$

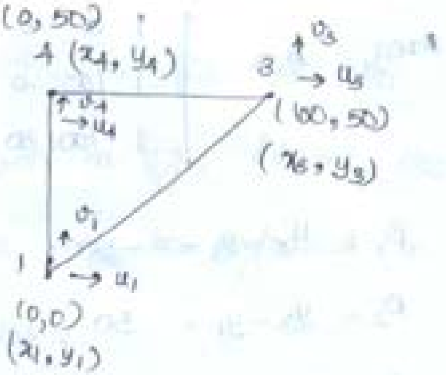
$$[B] = \frac{1}{100}\begin{bmatrix} -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 & 2 \\ 0 & -1 & -2 & 1 & 2 & 0 \end{bmatrix}$$

$$[B]^T = \frac{1}{100}\begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & -2 \\ 0 & -2 & 1 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{bmatrix}$$

$$[D] = \frac{E}{(1-\mu^2)}\begin{bmatrix} 1 & \mu & 0 \\ \mu & 1 & 0 \\ 0 & 0 & \frac{1-\mu}{2} \end{bmatrix} = 56 \times 10^3 \begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 1.5 \end{bmatrix}$$

$$[K]_1 = 14 \times 10^4 \begin{array}{c} \\ \left[\begin{array}{cccccc} u_1 & v_1 & u_2 & v_2 & u_3 & v_3 \\ 4 & 0 & -4 & 2 & 0 & -2 \\ 0 & 1.5 & 3 & -1.5 & -3 & 0 \\ -4 & 3 & 10 & -5 & -6 & 2 \\ 2 & -1.5 & -5 & 17.5 & 3 & -16 \\ 0 & -3 & -6 & 3 & 6 & 0 \\ -2 & 0 & 2 & -16 & 0 & 16 \end{array}\right] \end{array} \begin{array}{c} \\ u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{array}$$

consider element ②



$$[K]_2 = [B]^T [D] [B] A t$$

$$A = \frac{1}{2} \begin{vmatrix} 1 & 0 & 0 \\ 1 & 100 & 50 \\ 1 & 0 & 50 \end{vmatrix} = 2500 \text{ mm}^2$$

$\beta_1 = y_3 - y_4 = 0$; $\gamma_1 = x_4 - x_3 = +100$

$\beta_2 = y_4 - y_1 = 50$; $\gamma_2 = x_1 - x_4 = 0$

$\beta_3 = y_1 - y_3 = -50$; $\gamma_3 = x_3 - x_1 = 100$;

$$\therefore [B] = \frac{1}{100} \begin{bmatrix} 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & -2 & 0 & 0 & 0 & 2 \\ -2 & 0 & 0 & 1 & 2 & -1 \end{bmatrix}$$

$$[B]^T = \frac{1}{100} \begin{bmatrix} 0 & 0 & -2 \\ 0 & -2 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 2 \\ 0 & 2 & -1 \end{bmatrix}$$

$$[D] = \frac{E}{(1-\mu^2)} \begin{bmatrix} 1 & \mu & 0 \\ \mu & 1 & 0 \\ 0 & 0 & \frac{1-\mu}{2} \end{bmatrix} = 56 \times 10^3 \begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 1.5 \end{bmatrix}$$

$$\therefore [K]_2 = 14 \times 10^4 \begin{array}{c} \begin{array}{cccccc} u_1 & v_1 & u_3 & v_3 & u_4 & v_4 \end{array} \\ \left[\begin{array}{cccccc} 6 & 0 & 0 & -3 & -6 & 3 \\ 0 & 16 & -2 & 0 & 2 & -16 \\ 0 & -2 & 4 & 0 & -4 & 2 \\ -3 & 0 & 0 & 1.5 & 3 & -1.5 \\ -6 & 2 & -4 & 3 & 10 & -5 \\ 3 & -16 & 2 & -1.5 & -5 & 17.5 \end{array}\right] \end{array} \begin{array}{c} \\ u_1 \\ v_1 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{array}$$

Assembling the stiffness matrix, [K]

$$[K] = 14\times10^4 \begin{array}{c|cccccccc|c} & u_1 & v_1 & u_2 & v_2 & u_3 & v_3 & u_4 & v_3 & \\ \hline & 10 & 0 & -4 & 2 & 0 & -5 & -6 & 3 & u_1 \\ & 0 & 17.5 & 3 & -1.5 & -5 & 0 & 2 & -16 & v_1 \\ & -4 & 3 & 10 & -5 & -6 & 2 & 0 & 0 & u_2 \\ & 2 & -1.5 & -5 & 17.5 & 3 & -16 & 0 & 0 & v_2 \\ & 0 & -5 & -6 & 3 & 10 & 0 & -4 & 2 & u_3 \\ & -5 & 0 & 2 & -16 & 0 & 17.5 & 3 & -1.5 & v_3 \\ & -6 & 2 & 0 & 0 & -4 & 3 & 10 & -5 & u_4 \\ & 3 & -16 & 0 & 0 & 2 & -1.5 & -5 & 17.5 & v_4 \end{array}$$

Applying Boundary conditions;

$u_1 = v_1 = v_2 = u_4 = v_4 = 0;$

$F_{2x} = 1000$ N; $F_{3x} = 1000$ N; $F_{3y} = -10000$ N

Neglecting 1st, 2nd; 4th, 7th and 8th rows and columns in the finite element Equation.

$$14\times10^4 \begin{bmatrix} 10 & -6 & 2 \\ -6 & 10 & 0 \\ 2 & 0 & 17.5 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \\ v_3 \end{Bmatrix} = \begin{Bmatrix} 1000 \\ 1000 \\ -10000 \end{Bmatrix}$$

Solving we get,

$u_2 = 0.0032$ mm;

$u_3 = 0.0026$ mm;

$v_3 = -0.0044$ mm.

To find Element stresses,

$$\{\sigma\} = \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = [D]\,[B]\,\{\delta\} \quad ; \quad \delta = \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix}$$

For element ①

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}_1 = 56\times10^3 \begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 1.5 \end{bmatrix} \frac{1}{100} \begin{bmatrix} -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 & 2 \\ 0 & -1 & -2 & 1 & 2 & 0 \end{bmatrix} \times \begin{Bmatrix} 0 \\ 0 \\ 0.0032 \\ 0 \\ 0.0026 \\ -0.0044 \end{Bmatrix}$$

Solving,

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}_1 = \begin{Bmatrix} 2.24 \\ -17.92 \\ -1.008 \end{Bmatrix} N/mm^2$$

Similarly for Element ②

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}_2 = 56\times10^3 \begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 1.5 \end{bmatrix} \frac{1}{100} \begin{bmatrix} 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & -2 & 0 & 0 & 0 & 2 \\ -2 & 0 & 0 & 1 & 2 & -1 \end{bmatrix} \times \begin{Bmatrix} 0 \\ 0 \\ 0.0026 \\ -0.0044 \\ 0 \\ 0 \end{Bmatrix}$$

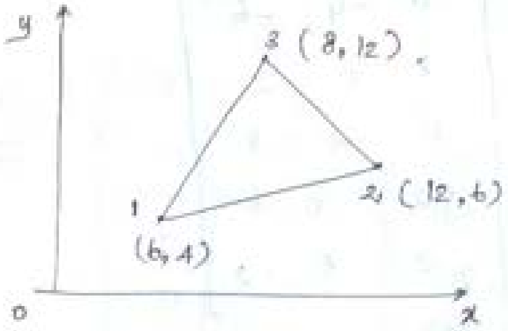
Solving, we get

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}_2 = \begin{Bmatrix} 5.824 \\ 1.456 \\ -2.696 \end{Bmatrix} N/mm^2$$

The element shown in fig. is subjected to a temperature change $10^\circ c$. Find the load due to temperature change.

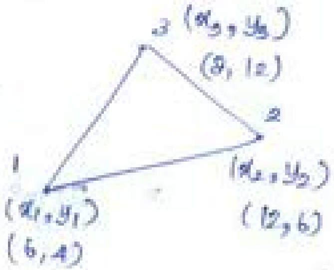
Take, $E = 200$ GPa; $\mu = 0.3$; $t = 2$ mm; $\alpha = 7 \times 10^{-6} / ^\circ c$

Assume plane stress conditions: The co-ordinates are in mm



Solution:

Nodal co-ordinates are,



$x_1 = 6mm$; $y_1 = 4mm$;

$x_2 = 12mm$; $y_2 = 6mm$

$x_3 = 8mm$; $y_3 = 12mm$.

$\Delta T = 10^\circ c$; $\alpha = 7 \times 10^{-6}$ $/^\circ c$; $\mu = 0.3$; $t = 2mm$.

Thermal load, $\{F_0\} = [B]^T [D] \{e_0\} A t$

$$\underset{\text{Thermal strain}}{\downarrow}$$

$$[B] = \frac{1}{2A} \begin{bmatrix} \beta_1 & 0 & \beta_2 & 0 & \beta_3 & 0 \\ 0 & \gamma_1 & 0 & \gamma_2 & 0 & \gamma_3 \\ \gamma_1 & \beta_1 & \gamma_2 & \beta_2 & \gamma_3 & \beta_3 \end{bmatrix}$$

$$A = \frac{1}{2} \begin{vmatrix} 1 & 6 & 4 \\ 1 & 12 & 6 \\ 1 & 8 & 12 \end{vmatrix} = 22 mm^2$$

$\beta_1 = y_2 - y_3 = -6$; $\gamma_1 = x_3 - x_2 = -4$

$\beta_2 = y_3 - y = 8$; $\gamma_2 = x_1 - x_3 = -2$

$\beta_3 = y_1 - y_2 = -2$; $\gamma_3 = x_2 - x_1 = 6$

$$\therefore [B] = \frac{1}{44}\begin{bmatrix} -6 & 0 & 8 & 0 & -2 & 0 \\ 0 & -4 & 0 & -2 & 0 & 6 \\ -4 & -6 & -2 & 8 & 6 & -2 \end{bmatrix}$$

$$[B]^T = \frac{1}{44}\begin{bmatrix} -6 & 0 & -4 \\ 0 & -4 & -6 \\ 8 & 0 & -2 \\ 0 & -2 & 8 \\ -2 & 0 & 6 \\ 0 & 6 & -2 \end{bmatrix}$$

$$[D] = \frac{E}{(1-\mu^2)}\begin{bmatrix} 1 & \mu & 0 \\ \mu & 1 & 0 \\ 0 & 0 & \frac{1-\mu}{2} \end{bmatrix}$$

$$= \frac{2\times10^6}{91}\begin{bmatrix} 10 & 3 & 0 \\ 3 & 10 & 0 \\ 0 & 0 & 3.5 \end{bmatrix}$$

Plane stress condition,

$$\{e_0\} = \begin{Bmatrix} \alpha\Delta T \\ \alpha\Delta T \\ 0 \end{Bmatrix} = \begin{Bmatrix} 7\times10^{-5} \\ 7\times10^{-5} \\ 0 \end{Bmatrix}$$

$\therefore$ Thermal load, $\{F_0\}$

$$\begin{Bmatrix} F_{01x} \\ F_{01y} \\ F_{02x} \\ F_{02y} \\ F_{03x} \\ F_{03y} \end{Bmatrix} = \frac{1}{44}\begin{bmatrix} -6 & 0 & -4 \\ 0 & -4 & -6 \\ 8 & 0 & -2 \\ 0 & -2 & 8 \\ -2 & 0 & 6 \\ 0 & 6 & -2 \end{bmatrix} \times \frac{2\times10^6}{91}\begin{bmatrix} 10 & 3 & 0 \\ 3 & 10 & 0 \\ 0 & 0 & 3.5 \end{bmatrix}\begin{Bmatrix} 7\times10^{-5} \\ 7\times10^{-5} \\ 0 \end{Bmatrix}$$

$$\times (22\times2)$$

$$= \frac{2\times10^{6}\times7\times10^{-5}}{91}\begin{bmatrix} -60 & -18 & -14 \\ -12 & -40 & -21 \\ 80 & 24 & -7 \\ -6 & -20 & 28 \\ -20 & -6 & 21 \\ 18 & 60 & -7 \end{bmatrix}\begin{Bmatrix} 1 \\ 1 \\ 0 \end{Bmatrix}$$

$$= \frac{140}{91}\begin{Bmatrix} -78 \\ -52 \\ 104 \\ -26 \\ -26 \\ 78 \end{Bmatrix} = \begin{Bmatrix} -120 \\ -80 \\ 160 \\ -40 \\ -40 \\ 120 \end{Bmatrix} N$$

loads due to temperature change,

$F_{01x} = -120$ N ; $F_{02y} = -40$ N

$F_{01y} = -80$ N ; $F_{03x} = -40$ N

$F_{02x} = 160$ N ; $F_{03y} = 120$ N

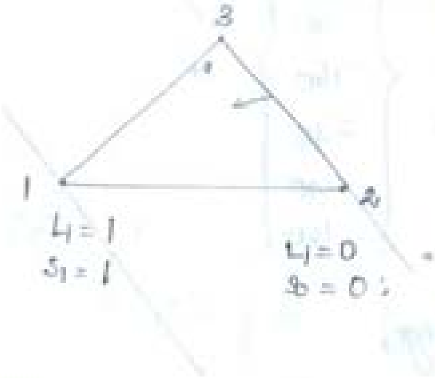
–ve sign → direction opposite to global axis direction.

LST– linear strain Triangle.

Lagrange Triangular element.

Interpolation function.

Linear Element.



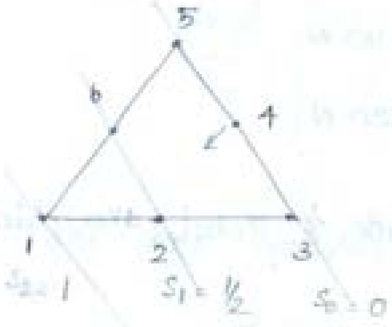
Shape function, $\psi_1 = \frac{L_1 - s_0}{s_1 - s_0} = \frac{L_1 - 0}{1 - 0} = L_1$

Illy

$\psi_2 = L_2$;

$\psi_3 = L_3$

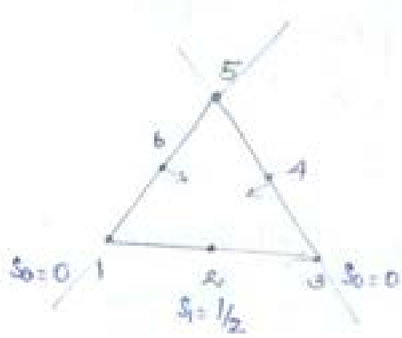
Quadratic Element.



$\psi_1 = \frac{L_1 - s_0}{s_2 - s_0} \times \frac{L_1 - s_1}{s_2 - s_1}$

$\psi_1 = \frac{L_1}{1} \times \frac{L_1 - 1/2}{1 - 1/2}$

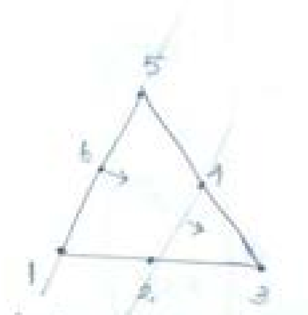
$\psi_1 = L_1 \; (2L_1 - 1)$



$\psi_2 = \frac{L_2 - s_0}{s_1 - s_0} \times \frac{L_1 - s_0}{S_1 - s_0}$

$= \frac{L_2}{1/2} \times \frac{L_1}{1/2}$

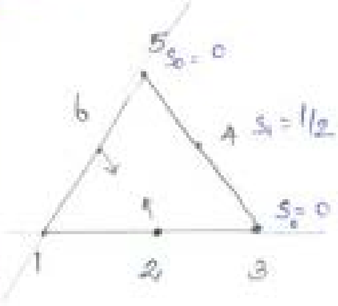
$\psi_2 = 4 L_2 L_1$



$\psi_3 = \frac{L_2 - s_0}{s_2 - s_0} \times \frac{L_2 - s_1}{s_2 - s_1}$

$= \frac{L_2}{1} \times \frac{L_2 - 1/2}{1 - 1/2}$

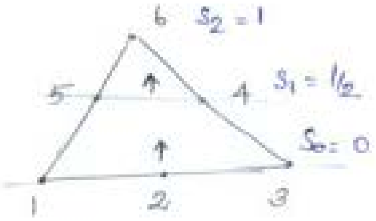
$\psi_3 = L_2 \; (2L_2 - 1)$



$$\psi_4 = \frac{L_2 - s_0}{s_1 - s_0} \times \frac{L_3 - s_0}{s_1 - s_0}$$

$$= \frac{L_2}{1/2} \times \frac{L_3}{1/2}$$

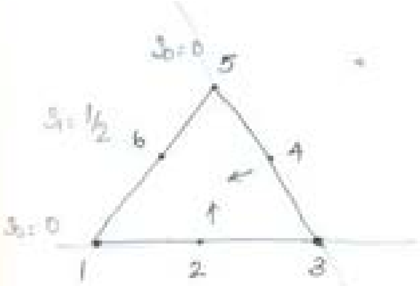
$$\psi_4 = 4 L_2 L_3 .$$



$$\psi_5 = \frac{L_3 - s_0}{s_2 - s_0} \times \frac{L_3 - s_1}{s_2 - s_1}$$

$$= \frac{L_3}{1} \times \frac{L_3 - 1/2}{1 - 1/2}$$

$$\psi_5 = L_3 (2 L_3 - 1)$$



$$\psi_6 = \frac{L_1 - s_0}{s_1 - s_0} \times \frac{L_3 - s_0}{s_1 - s_0}$$

$$= \frac{L_1}{1/2} \times \frac{L_3}{1/2}$$

$$\psi_6 = 4 L_1 L_3 .$$

UNIT - IV

TWO DIMENSIONAL VECTOR VARIABLE PROBLEMS.

Equations of elasticity - Plane stress, Plane strain and axisymmetric problems - Body forces and temperature effects - stress calculations - Plate and shell elements.

—x—

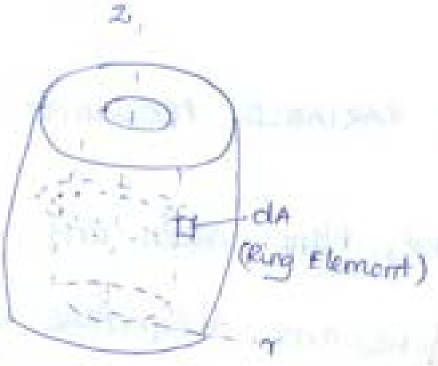
Axi-symmetric problems:

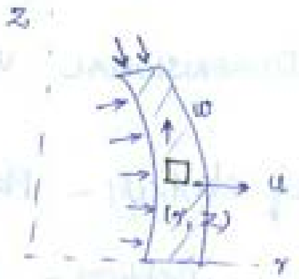
Axisymmetric solids (or) solids of revolutions, all the 3 dimensions are comparatively large, they may be analysed using 2 dimensional techniques due to their axial symmetry.

Ex: Pressure vessels, cylinders loaded by uniform internal or External Pressures, Turbine discs, flywheels etc.

* usually z-axis may be considered as the axis of symmetry and any point in the plane is represented by the polar coordinates (r, θ)

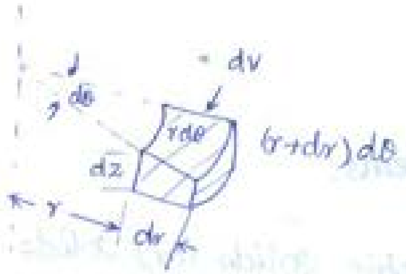
* Rotational angle θ is considered, deformation and stresses are independent of co-ordinate θ.





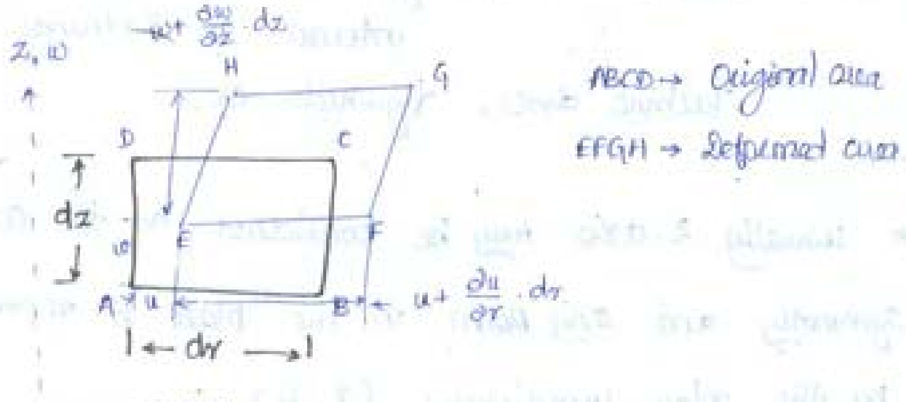
Elasticity relations for axisymmetric Elements:

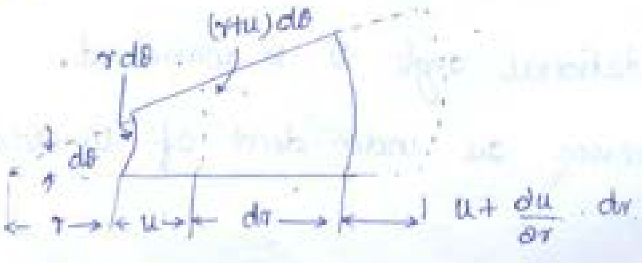
consider an elemental volume of the ring element as shown in fig.



The strains and stresses of different categories are derived as follows:

For this element, the deformation can be specified in r-θ as well as r-z planes.





a) radial strain, $e_r = \dfrac{\left(u + \frac{\partial u}{\partial r} \cdot dr\right) - u}{dr} = \dfrac{\partial u}{\partial r}$

b) Tangential strain, (or) circumferential strain $e_\theta = \dfrac{(r+u)\,d\theta - r\,d\theta}{r\,d\theta} = \dfrac{u}{r}$

c) Axial (or) longitudinal strain, $e_z = \dfrac{\left(w + \frac{\partial w}{\partial z} \cdot dz\right) - w}{dz} = \dfrac{\partial w}{\partial z}$

d) shear strain, $\gamma_{rz} = \dfrac{\left(u + \frac{\partial u}{\partial z} dz\right) - u}{dz} + \dfrac{\left(w + \frac{\partial w}{\partial r} dr\right) - w}{dr}$

$$= \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}$$

strains in matrix form,

$$\{e\} = \begin{Bmatrix} e_r \\ e_\theta \\ e_x \\ \gamma_{xz} \end{Bmatrix} = \begin{Bmatrix} \partial u/\partial r \\ u/r \\ \partial w/\partial z \\ \frac{\partial u}{\partial r} + \frac{\partial w}{\partial r} \end{Bmatrix}$$

Ily
stresses in matrix form,

$$\{\sigma\} = \begin{Bmatrix} \sigma_r \\ \sigma_\theta \\ \sigma_z \\ \tau_{rz} \end{Bmatrix}$$

σ_r → Radial stress
σ_θ → Tangential / circumferential stress
σ_z → Axial / longitudinal stress
τ_{rz} → shear stress.

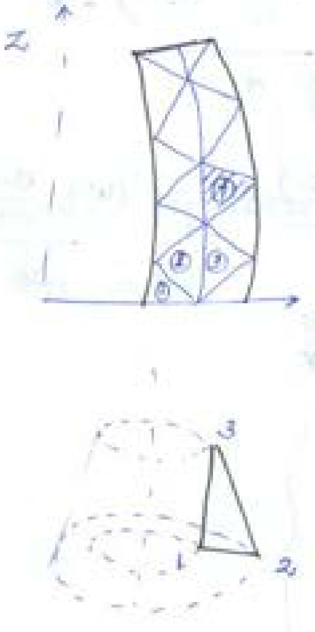
As per. Hooke's Law:

$$\{\sigma\} = [D]\{e\}$$

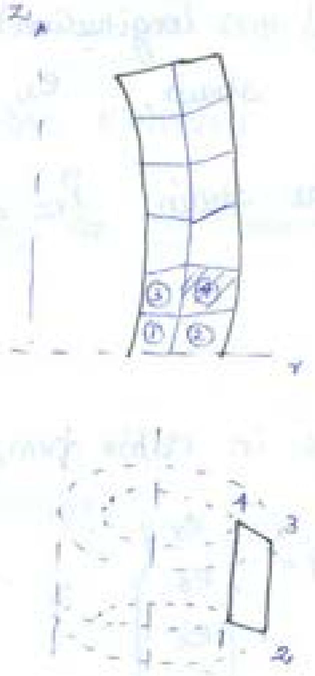
[D] → stress - strain relationship (or) constitutive matrix

Finite Element modeling for Axi-symmetric Solid.

The solid of revolution is completely represented by revolving the triangular / Quadrilateral element about the z-axis.



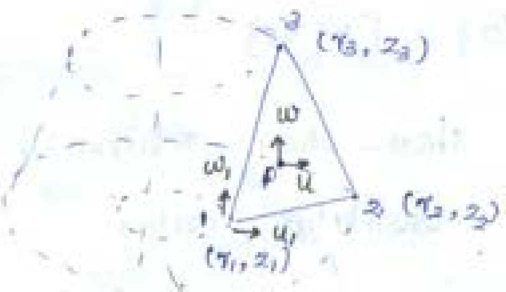
Ring shaped solid



Ring shape solid.

Derivation of shape function for Axisymmetric Triangular Element.

Consider an axisymmetric triangular element with nodes 1, 2, 3 as shown in fig.



let; u_1, w_1; u_2, w_2; u_3, w_3 : displacements along radial direction r & axial direction z.

Nodal displacement, in matrix form, $\{2\} = \begin{Bmatrix} u_1 \\ w_1 \\ u_2 \\ w_2 \\ u_3 \\ w_3 \end{Bmatrix}$

The displacement function at the

point P, in polynomial series

$u(r, z) = a_1 + a_2 r + a_3 z$ — (1) $a \rightarrow$ Polynomial co-efficients.

$w(r, z) = a_4 + a_5 r + a_6 z$ — (2)

Applying the nodal conditions,

$u = u_1$; $w = w_1$ at (r_1, z_1)

$u = u_2$; $w = w_2$ at (r_2, z_2)

$u = u_3$; $w = w_3$ at (r_3, z_3)

∴ Nodal displacements,

$u_1 = a_1 + a_2 r_1 + a_3 z_1$

$u_2 = a_1 + a_2 r_2 + a_3 z_2$

$u_3 = a_1 + a_2 r_3 + a_3 z_3$

^{lly} for the w_1, w_2 & w_3.

The above equation in matrix form,

$$\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{bmatrix} 1 & r_1 & z_1 \\ 1 & r_2 & z_2 \\ 1 & r_3 & z_3 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix}$$

$$\begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} = \begin{bmatrix} 1 & r_1 & z_1 \\ 1 & r_2 & z_2 \\ 1 & r_3 & z_3 \end{bmatrix}^{-1} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} \quad — (3)$$

$\{a\} = [D]^{-1} \{u\}$; $[D] \rightarrow$ co-ordinate matrix.

$[D]^{-1} = \frac{[C]^T}{|D|}$; $[C] \rightarrow$ cofactor matrix of $[D]$.

co-factor matrix, $[C] = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$

$$[D] = \begin{bmatrix} 1 & r_1 & z_1 \\ 1 & r_2 & z_2 \\ 1 & r_3 & z_3 \end{bmatrix}$$

$$[C] = \begin{bmatrix} (r_2 z_3 - r_3 z_2) & z_2 - z_3 & r_3 - r_2 \\ (r_3 z_1 - r_1 z_3) & z_3 - z_1 & r_1 - r_3 \\ (r_1 z_2 - r_2 z_1) & z_1 - z_2 & r_2 - r_1 \end{bmatrix}$$

$$[C]^T = \begin{bmatrix} r_2 z_3 - r_3 z_2 & r_3 z_1 - r_1 z_3 & r_1 z_2 - r_2 z_1 \\ z_2 - z_3 & z_3 - z_1 & z_1 - z_2 \\ r_3 - r_2 & r_1 - r_3 & r_2 - r_1 \end{bmatrix}$$

$$|D| = 1(r_2 z_3 - r_3 z_2) - r_1 (z_3 - z_2) + z_1 (r_3 - r_2)$$

$$= 2A.$$

Area of the triangle, $A = \frac{1}{2} \begin{vmatrix} 1 & r_1 & z_1 \\ 1 & r_2 & z_2 \\ 1 & r_3 & z_3 \end{vmatrix}$

Substituting the values in Equ (2)

$$\begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} = \frac{1}{2A} \begin{bmatrix} r_2 z_3 - r_3 z_2 & r_3 z_1 - r_1 z_3 & r_1 z_2 - r_2 z_1 \\ z_2 - z_3 & z_3 - z_1 & z_1 - z_2 \\ r_3 - r_2 & r_1 - r_3 & r_2 - r_1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

$$= \frac{1}{2A} \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

where,

$\alpha_1 = r_2 z_3 - r_3 z_2$; $\alpha_2 = r_3 z_1 - r_1 z_3$; $\alpha_3 = r_1 z_2 - r_2 z_1$

$\beta_1 = z_2 - z_3$; $\beta_2 = z_3 - z_1$; $\beta_3 = z_1 - z_2$

$\gamma_1 = r_3 - r_2$; $\gamma_2 = r_1 - r_3$; $\gamma_3 = r_2 - r_1$

Equ (1) in matrix form, $u(r,z) = \begin{bmatrix} 1 & r & z \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix}$

$$u(r,z) = \begin{bmatrix} 1 & r & z \end{bmatrix} \frac{1}{2A} \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

$$= \frac{1}{2A} \begin{bmatrix} \alpha_1 + \beta_1 r + \gamma_1 z & \alpha_2 + \beta_2 r + \gamma_2 z & \alpha_3 + \beta_3 r + \gamma_3 z \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

$$= \begin{bmatrix} N_1 & N_2 & N_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

(or)

$$u = N_1 u_1 + N_2 u_2 + N_3 u_3.$$

Similarly,

$$w = N_1 w_1 + N_2 w_2 + N_3 w_3$$

where,

$$\left. \begin{aligned} N_1 &= \alpha_1 + \beta_1 r + \gamma_1 z / 2A \\ N_2 &= \alpha_2 + \beta_2 r + \gamma_2 z / 2A \\ N_3 &= \alpha_3 + \beta_3 r + \gamma_3 z / 2A \end{aligned} \right\} \text{shape functions}$$

Assembling the Equ (1) & (2) in matrix form,

$$\delta(r,z)_p = \begin{Bmatrix} u(r,z) \\ w(r,z) \end{Bmatrix}_p = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ w_1 \\ u_2 \\ w_2 \\ u_3 \\ w_3 \end{Bmatrix}$$

$$\delta_p = [N]\{\delta\}$$

δ_p → Displacements at the point $P = \begin{Bmatrix} u \\ w \end{Bmatrix}_p$

$\{\delta\}$ = Nodal displacement vector

$[N]$ = shape functions matrix.

Stress – strain Relationship / constitutive Matrix for Axisymmetric Triangular Element.

The stress-strain relationship expressions of three dimensional solid for axisymmetric element by replacing x by r; y by θ; leaving z as it is.

3D-solid.	Axisymmetric solid
$e_x = \frac{\sigma_x}{E} - \mu \frac{\sigma_y}{E} - \mu \cdot \frac{\sigma_z}{E}$;	$e_r = \frac{\sigma_r}{E} - \mu \cdot \frac{\sigma_\theta}{E} - \mu \cdot \frac{\sigma_z}{E}$
$e_y = \frac{\sigma_x}{E}(-\mu) + \frac{\sigma_y}{E} - \mu \cdot \frac{\sigma_z}{E}$;	$e_\theta = -\mu \frac{\sigma_r}{E} + \frac{\sigma_\theta}{E} - \mu \frac{\sigma_z}{E}$
$e_z = -\mu \cdot \frac{\sigma_x}{E} - \mu \cdot \frac{\sigma_y}{E} + \frac{\sigma_z}{E}$;	$\sigma_z = -\mu \cdot \frac{\sigma_r}{E} - \mu \cdot \frac{\sigma_\theta}{E} + \frac{\sigma_z}{E}$
$\gamma_{xz} = \frac{2(1+\mu)}{E} \tau_{xz}$;	$\gamma_{rz} = \frac{2(1+\mu)}{E} \tau_{rz}$.

Solving the above set of equations, we get stresses.

Radial stress, $\sigma_r = \frac{E}{(1+\mu)(1-2\mu)} \left[(1-\mu) e_r + \mu \cdot e_\theta + \mu \cdot e_z \right]$

Circumferential stress, $\sigma_\theta = \frac{E}{(1+\mu)(1-2\mu)} \left[\mu \cdot e_r + (1-\mu) e_\theta + \mu \cdot e_z \right]$

Axial stress, $\sigma_z = \frac{E}{(1+\mu)(1-2\mu)} \left[\mu \cdot e_r + \mu \cdot e_\theta + (1-\mu) e_z \right]$

Shear stress, $\tau_{rz} = \frac{E}{(1+\mu)(1-2\mu)} \left[\frac{1-2\mu}{2} \right] \gamma_{rz}$

In matrix form,

$$\begin{Bmatrix} \sigma_r \\ \sigma_\theta \\ \sigma_z \\ \tau_{rz} \end{Bmatrix} = \frac{E}{(1+\mu)(1-2\mu)} \begin{bmatrix} 1-\mu & \mu & \mu & 0 \\ \mu & 1-\mu & \mu & 0 \\ \mu & \mu & 1-\mu & 0 \\ 0 & 0 & 0 & \frac{1-2\mu}{2} \end{bmatrix} \begin{Bmatrix} e_r \\ e_\theta \\ e_z \\ \gamma_{rz} \end{Bmatrix}$$

stress - strain Relationship matrix, [Plane strain]

$$[D] = \frac{E}{(1+\mu)(1-2\mu)} \begin{bmatrix} 1-\mu & \mu & \mu & 0 \\ \mu & 1-\mu & \mu & 0 \\ \mu & \mu & 1-\mu & 0 \\ 0 & 0 & 0 & \frac{1-2\mu}{2} \end{bmatrix}$$

strain - displacement / Gradient Matrix.

displacement functions,

$$\delta(r,z) = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ w_1 \\ u_2 \\ w_2 \\ u_3 \\ w_3 \end{Bmatrix}$$

$$u = N_1 u_1 + N_2 u_2 + N_3 u_3$$

$$w = N_1 w_1 + N_2 w_2 + N_3 w_3.$$

strains for axisymmetric element,

$$\{e\} = \begin{Bmatrix} e_r \\ e_\theta \\ e_z \\ \gamma_{rz} \end{Bmatrix} = \begin{Bmatrix} \partial u/\partial r \\ u/r \\ \partial w/\partial z \\ \partial u/\partial z + \partial w/\partial r \end{Bmatrix}$$

Radial strain, $e_r = \frac{\partial u}{\partial r} = \frac{\partial N_1}{\partial r} u_1 + \frac{\partial N_2}{\partial r} u_2 + \frac{\partial N_3}{\partial r} u_3$

circumf strain, $e_\theta = \frac{u}{r} = \frac{N_1}{r} u_1 + \frac{N_2}{r} u_2 + \frac{N_3}{r} u_3$

Axial strain, $e_z = \frac{\partial w}{\partial z} = \frac{\partial N_1}{\partial z} w_1 + \frac{\partial N_2}{\partial z} w_2 + \frac{\partial N_3}{\partial z} w_3$

Shear strain, $\gamma_{rz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}$

$$= \frac{\partial N_1}{\partial z} u_1 + \frac{\partial N_2}{\partial z} u_2 + \frac{\partial N_3}{\partial z} u_3 + \frac{\partial N_1}{\partial r} w_1 + \frac{\partial N_2}{\partial r} w_2 + \frac{\partial N_3}{\partial r} w_3$$

In matrix form.

$$\begin{Bmatrix} e_r \\ e_\theta \\ e_z \\ \gamma_{rz} \end{Bmatrix} = \begin{bmatrix} \frac{\partial N_1}{\partial r} & 0 & \frac{\partial N_2}{\partial r} & 0 & \frac{\partial N_3}{\partial r} & 0 \\ \frac{N_1}{r} & 0 & \frac{N_2}{r} & 0 & \frac{N_3}{r} & 0 \\ 0 & \frac{\partial N_1}{\partial z} & 0 & \frac{\partial N_2}{\partial z} & 0 & \frac{\partial N_3}{\partial z} \\ \frac{\partial N_1}{\partial z} & \frac{\partial N_1}{\partial r} & \frac{\partial N_2}{\partial z} & \frac{\partial N_3}{\partial r} & \frac{\partial N_3}{\partial z} & \frac{\partial N_3}{\partial r} \end{bmatrix} \begin{Bmatrix} u_1 \\ w_1 \\ u_2 \\ w_2 \\ u_3 \\ w_3 \end{Bmatrix}$$

$$N_1 = \frac{1}{2A} (\alpha_1 + \beta_1 r + \gamma_1 z)$$

$$N_2 = \frac{1}{2A} (\alpha_2 + \beta_2 r + \gamma_2 z)$$

$$N_3 = \frac{1}{2A} (\alpha_3 + \beta_3 r + \gamma_3 z)$$

$$\frac{\partial N_1}{\partial r} = \frac{\beta_1}{2A} ; \quad \frac{\partial N_2}{\partial r} = \frac{\beta_2}{2A} ; \quad \frac{\partial N_3}{\partial r} = \frac{\beta_3}{2A}$$

$$\frac{N_1}{r} = \frac{1}{2A} \left[\frac{\alpha_1}{r} + \beta_1 + \frac{\gamma_1 z}{r} \right]$$

$$\frac{N_2}{r} = \frac{1}{2A} \left[\frac{\alpha_2}{r} + \beta_2 + \frac{\gamma_2 z}{r} \right]$$

$$\frac{N_3}{r} = \frac{1}{2A} \left[\frac{\alpha_3}{r} + \beta_3 + \frac{\gamma_3 z}{r} \right]$$

$$\frac{\partial N_1}{\partial z} = \frac{\gamma_1}{2A} ; \quad \frac{\partial N_2}{\partial z} = \frac{\gamma_2}{2A} ; \quad \frac{\partial N_3}{\partial z} = \frac{\gamma_3}{2A}$$

Substituting the values in the above equation of matrix

$$\begin{Bmatrix} e_r \\ e_\theta \\ e_z \\ \gamma_{rz} \end{Bmatrix} = \frac{1}{2A} \begin{bmatrix} \beta_1 & 0 & \beta_2 & 0 & \beta_3 & 0 \\ \frac{\alpha_1+\beta_1 r+\gamma_1 z}{r} & 0 & \frac{\alpha_2+\beta_2 r+\gamma_2 z}{r} & 0 & \frac{\alpha_3+\beta_3 r+\gamma_3 z}{r} & 0 \\ 0 & \gamma_1 & 0 & \gamma_2 & 0 & \gamma_3 \\ \gamma_1 & \beta_1 & \gamma_2 & \beta_2 & \gamma_3 & \beta_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ w_1 \\ u_2 \\ w_2 \\ u_3 \\ w_3 \end{Bmatrix}$$

$\{e\} = [B]\{u\}$

$[B]$ → strain displacement / Gradient Matrix.

Stress - displacement Relations:

For an Axisymmetric triangular element.

Stress, $\{\sigma\} = [D]\{e\}$

Strain, $\{e\} = [B]\{\delta\}$

$\therefore$ Stress, $\{\sigma\} = [D][B]\{\delta\}$

Stiffness matrix for axisymmetric triangular Element.

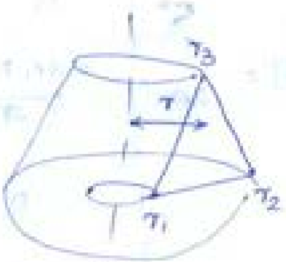
Stiffness matrix, $[K] = \int_v [B]^T [D][B] \, dv$

$dv = 2\pi r . A$

$\therefore [K] = 2\pi r . A \; [B]^T [D][B]$

where,

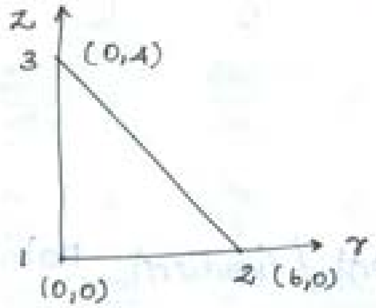
$r = \frac{r_1 + r_2 + r_3}{3}$



Determine the element stiffness matrix for the axisymmetric triangular element shown in fig.

Assume, $E = 2 \times 10^5 \ N/mm^2$; $\mu = 0.3$;

The co-ordinates are in cm.



Solution:-

The nodal co-ordinates of the axisymmetric element are,

$r_1 = 0 \ cm$; $r_2 = 6 \ cm$; $r_3 = 0 \ cm$

$z_1 = 0 \ cm$; $z_2 = 0 \ cm$; $z_3 = 4 \ cm$.

$E = 2 \times 10^5 \ N/mm^2 = 2 \times 10^7 \ N/cm^2$.

Stiffness matrix, $[K] = 2\pi r . A \ [B]^T [D] [B]$

$$\text{Area, } A = \frac{1}{2} \begin{vmatrix} 1 & r_1 & z_1 \\ 1 & r_2 & z_2 \\ 1 & r_3 & z_3 \end{vmatrix} = \frac{1}{2} \begin{vmatrix} 1 & 0 & 0 \\ 1 & 6 & 0 \\ 0 & 0 & 4 \end{vmatrix}$$

$$= \frac{24}{2} = 12 cm^2 ;$$

strain-displacement matrix,

$$[B] = \frac{1}{2A} \begin{bmatrix} \beta_1 & 0 & \beta_2 & 0 & \beta_3 & 0 \\ \frac{\alpha_1 + \beta_1 r + \gamma_1 z}{r} & 0 & \frac{\alpha_2 + \beta_2 r + \gamma_2 z}{r} & 0 & \frac{\alpha_3 + \beta_3 r + \gamma_3 z}{r} & 0 \\ 0 & \gamma_1 & 0 & \gamma_2 & 0 & \gamma_3 \\ \gamma_1 & \beta_1 & \gamma_2 & \beta_2 & \gamma_3 & \beta_3 \end{bmatrix}$$

$\alpha_1 = r_2 z_3 - r_3 z_2 = (6 \times 4) - 0 = 24;$

$\alpha_2 = r_3 z_1 - r_1 z_3 = 0;$

$\alpha_3 = r_1 z_2 - r_2 z_1 = 0;$

$\beta_1 = z_2 - z_3 = -4;$ $\gamma_1 = r_3 - r_2 = -6$

$\beta_2 = z_3 - z_1 = 4$ $\gamma_2 = r_1 - r_3 = 0$

$\beta_3 = z_1 - z_2 = 0;$ $\gamma_3 = r_2 - r_1 = 6$

co-ordinates, $r = \frac{r_1 + r_2 + r_3}{3} = \frac{6}{3} = 2,$

$z = \frac{z_1 + z_2 + z_3}{3} = \frac{4}{3} = 1.33$

$$\frac{\alpha_1 + \beta_1 r + \gamma_1 z}{r} = \frac{24 - (4 \times 2) - (6 \times 1.33)}{2} = 4$$

$$\frac{\alpha_2 + \beta_2 r + \gamma_2 z}{r} = \frac{0 + (4 \times 2) + 0}{2} = 4$$

$$\frac{\alpha_3 + \beta_3 r + \gamma_3 z}{r} = \frac{(0 + 0 + 6 \times 1.33)}{2} = 4.$$

$$\therefore [B] = \frac{1}{24} \begin{bmatrix} -4 & 0 & 4 & 0 & 0 & 0 \\ 4 & 0 & 4 & 0 & 4 & 0 \\ 0 & -6 & 0 & 0 & 0 & 6 \\ -6 & -4 & 0 & 4 & 6 & 0 \end{bmatrix}$$

$$[B]^T = \frac{1}{24} \begin{bmatrix} -4 & 4 & 0 & -6 \\ 0 & 0 & -6 & -4 \\ 4 & 4 & 0 & 0 \\ 0 & 0 & 0 & 4 \\ 0 & 4 & 0 & 6 \\ 0 & 0 & 6 & 0 \end{bmatrix}$$

stress-strain relationship matrix,

$$[D] = \frac{E}{(1+\mu)(1-2\mu)} \begin{bmatrix} 1-\mu & \mu & \mu & 0 \\ \mu & 1-\mu & \mu & 0 \\ \mu & \mu & 1-\mu & 0 \\ 0 & 0 & 0 & \frac{1-2\mu}{2} \end{bmatrix}$$

$$= \frac{2\times10^7}{1.3\times0.4} \begin{bmatrix} 0.7 & 0.3 & 0.3 & 0 \\ 0.3 & 0.7 & 0.3 & 0 \\ 0.3 & 0.3 & 0.7 & 0 \\ 0 & 0 & 0 & 0.2 \end{bmatrix}$$

$$[D] = \frac{10^8}{26} \begin{bmatrix} 7 & 3 & 3 & 0 \\ 3 & 7 & 3 & 0 \\ 3 & 3 & 7 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

$$[K] = 2\pi \times 2 \times 12 \times \frac{1}{24} \begin{bmatrix} -4 & 0 & 4 & 0 & 0 & 0 \\ 4 & 0 & 4 & 0 & 4 & 0 \\ 0 & -6 & 0 & 0 & 0 & 6 \\ -6 & -4 & 0 & 4 & 6 & 0 \end{bmatrix} \times$$

$$\frac{10^8}{26} \begin{bmatrix} 7 & 3 & 3 & 0 \\ 3 & 7 & 3 & 0 \\ 3 & 3 & 7 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \times \frac{1}{24} \begin{bmatrix} -4 & 4 & 0 & -6 \\ 0 & 0 & -6 & -4 \\ 4 & 4 & 0 & 0 \\ 0 & 0 & 0 & 4 \\ 0 & 4 & 0 & 6 \\ 0 & 0 & 6 & 0 \end{bmatrix}$$

$$\therefore [K] = 40.3\times10^5 \begin{bmatrix} 50 & 12 & 0 & -12 & -2 & 0 \\ 12 & 71 & -36 & -8 & -30 & -63 \\ 0 & -36 & 80 & 0 & 40 & 36 \\ -12 & -8 & 0 & 8 & 12 & 0 \\ -2 & -30 & 40 & 12 & 46 & 18 \\ 0 & -63 & 36 & 0 & 18 & 63 \end{bmatrix} \text{N/cm}$$

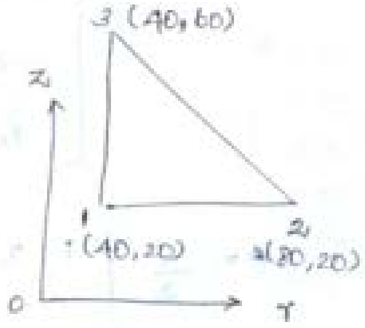
Calculate the element stresses for the axisymmetric element shown in fig. The nodal displacements are.

$u_1 = 0.02$ mm ; $w_1 = 0.03$ mm

$u_2 = 0.01$ mm ; $w_2 = 0.06$ mm

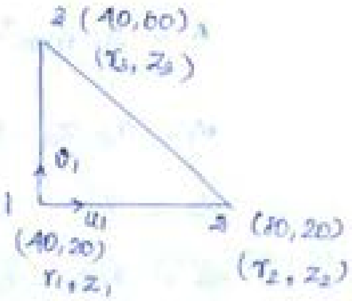
$u_3 = 0.04$ mm ; $w_3 = 0.01$ mm

Take, $E = 210$ GPa; $\mu = 0.25$



Solution :-

Nodal co-ordinates of the axisymmetric element are,



$r_1 = 40$ mm ; $z_1 = 20$ mm

$r_2 = 80$ mm ; $z_2 = 20$ mm

$r_3 = 40$ mm ; $z_3 = 60$ mm.

The element stresses. $\{\sigma\} = [D][B]\{\delta\}$

Stress-strain matrix,

$$[D] = \frac{E}{(1+\mu)(1-2\mu)} \begin{bmatrix} 1-\mu & \mu & \mu & 0 \\ \mu & 1-\mu & \mu & 0 \\ \mu & \mu & 1-\mu & 0 \\ 0 & 0 & 0 & \frac{1-2\mu}{2} \end{bmatrix}$$

$$= \frac{2.1\times10^5}{1.25\times0.5} \begin{bmatrix} 0.75 & 0.25 & 0.25 & 0 \\ 0.25 & 0.75 & 0.25 & 0 \\ 0.25 & 0.25 & 0.75 & 0 \\ 0 & 0 & 0 & 0.25 \end{bmatrix}$$

$$= 84\times10^3 \begin{bmatrix} 3 & 1 & 1 & 0 \\ 1 & 3 & 1 & 0 \\ 1 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Strain - displacement matrix,

$$[B] = \frac{1}{2A}\begin{bmatrix} \beta_1 & 0 & \beta_2 & 0 & \beta_3 & 0 \\ \frac{\alpha_1+\beta_1 r+\gamma_1 z}{r} & 0 & \frac{\alpha_2+\beta_2 r+\gamma_2 z}{r} & 0 & \frac{\alpha_3+\beta_3 r+\gamma_3 z}{r} & 0 \\ 0 & \gamma_1 & 0 & \gamma_2 & 0 & \gamma_3 \\ \gamma_1 & \beta_1 & \gamma_2 & \beta_2 & \gamma_3 & \beta_3 \end{bmatrix}$$

$$A = \frac{1}{2}\begin{vmatrix} 1 & r_1 & z_1 \\ 1 & r_2 & z_2 \\ 1 & r_3 & z_3 \end{vmatrix} = \frac{1}{2}\begin{vmatrix} 1 & 40 & 20 \\ 1 & 80 & 20 \\ 1 & 40 & 60 \end{vmatrix}$$

$A = 800\ mm^2$

$\alpha_1 = r_2 z_3 - r_3 z_2 = 4000$; $\beta_1 = z_2 - z_3 = -40$; $\gamma_1 = r_3 - r_2 = -40$

$\alpha_2 = r_3 z_1 - r_1 z_3 = -1600$; $\beta_2 = z_3 - z_1 = 40$: $\gamma_2 = r_1 - r_3 = 0$

$\alpha_3 = r_1 z_2 - r_2 z_1 = -800$; : $\beta_3 = z_1 - z_2 = 0$: $\gamma_3 = r_2 - r_1 = 40$

$$r = \frac{r_1 + r_2 + r_3}{3} = \frac{160}{3} ;$$

$$z = \frac{z_1 + z_2 + z_3}{3} = \frac{100}{3} ,$$

$$\frac{\alpha_1 + \beta_1 r + \gamma_1 z}{r} = 10 ; \quad \frac{\alpha_2 + \beta_2 r + \gamma_2 z}{r} = 10 ; \quad \frac{\alpha_3 + \beta_3 r + \gamma_3 z}{r} = 10 ;$$

$$\therefore [B] = \frac{1}{1600}\begin{bmatrix} -40 & 0 & 40 & 0 & 0 & 0 \\ 10 & 0 & 10 & 0 & 10 & 0 \\ 0 & -40 & 0 & 0 & 0 & 40 \\ -40 & -40 & 0 & 40 & 40 & 0 \end{bmatrix}$$

$$\{\delta\} = \begin{Bmatrix} u_1 \\ w_1 \\ u_2 \\ w_2 \\ u_3 \\ w_3 \end{Bmatrix} = \begin{Bmatrix} 0.02 \\ 0.03 \\ 0.01 \\ 0.06 \\ 0.04 \\ 0.01 \end{Bmatrix}$$

Stresses,

$$\{\sigma\} = \begin{Bmatrix} \sigma_r \\ \sigma_\theta \\ \sigma_z \\ \tau_{rz} \end{Bmatrix} = \frac{84\times10^3}{160}\begin{bmatrix} 3 & 1 & 1 & 0 \\ 1 & 3 & 1 & 0 \\ 1 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} -4 & 0 & 4 & 0 & 0 & 0 \\ 7 & 0 & 1 & 0 & 1 & 0 \\ 0 & -4 & 0 & 0 & 0 & 4 \\ -4 & -4 & 0 & 4 & 4 & 0 \end{bmatrix} \times \begin{Bmatrix} 0.02 \\ 0.03 \\ 0.01 \\ 0.06 \\ 0.04 \\ 0.01 \end{Bmatrix}$$

$$= 525\begin{Bmatrix} -0.13 \\ 0.09 \\ -0.21 \\ 0.20 \end{Bmatrix}$$

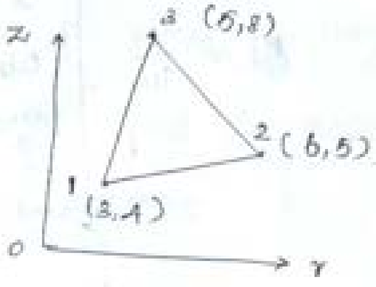
$$\begin{Bmatrix} \sigma_r \\ \sigma_\theta \\ \sigma_z \\ \tau_{rz} \end{Bmatrix} = \begin{Bmatrix} -68.25 \\ 47.25 \\ -110.25 \\ 105 \end{Bmatrix} \text{N/mm}^2$$

Compute the strain-displacement matrix for the axisymmetric triangular element shown in fig. Also determine the element strains. The nodal displacements are,

$u_1 = 0.002$; $w_1 = 0.001$

$u_2 = 0.001$; $w_2 = -0.004$

$u_3 = -0.003$ $w_3 = 0.007$



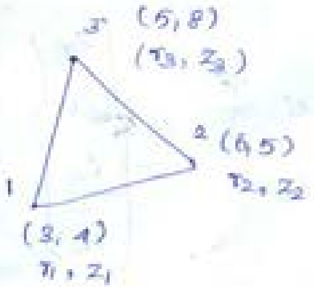
Solution:

The nodal co-ordinates are,

$r_1 = 3cm$; $z_1 = 4cm$

$r_2 = 6cm$ $z_2 = 5cm$

$r_3 = 5cm$; $z_3 = 8cm$;



$$r = \frac{r_1 + r_2 + r_3}{3} = 4.7$$

$$z = \frac{z_1 + z_2 + z_3}{3} = 5.7$$

Strain-displacement matrix.

$$[B] = \frac{1}{2A}\begin{bmatrix} \beta_1 & 0 & \beta_2 & 0 & \beta_3 & 0 \\ \frac{\alpha_1}{r} + \beta_1 + \frac{\gamma_1 z}{r} & 0 & \frac{\alpha_2}{r} + \beta_2 + \frac{\gamma_2 z}{r} & 0 & \frac{\alpha_3}{r} + \beta_3 + \frac{\gamma_3 z}{r} & 0 \\ 0 & \gamma_1 & 0 & \gamma_2 & 0 & \gamma_3 \\ \gamma_1 & \beta_1 & \gamma_2 & \beta_2 & \gamma_3 & \beta_3 \end{bmatrix}$$

$$A = \frac{1}{2}\begin{vmatrix} 1 & 3 & 4 \\ 1 & 6 & 5 \\ 1 & 5 & 8 \end{vmatrix} = 20\,cm^2;$$

$\alpha_1 = r_2 z_3 - r_3 z_2 = 28$; $\beta_1 = z_2 - z_3 = -3$; $\gamma_1 = r_3 - r_2 = -1$

$\alpha_2 = r_3 z_1 - r_1 z_3 = -4$; $\beta_2 = z_3 - z_1 = 4$; $\gamma_2 = r_1 - r_3 = -2$

$\alpha_3 = r_1 z_2 - r_2 z_1 = -9$; $\beta_3 = z_1 - z_2 = -1$; $\gamma_3 = r_2 - r_1 = 3$

strain - displacement matrix,

$$[B] = \frac{1}{10}\begin{bmatrix} -3 & 0 & 4 & 0 & -1 & 0 \\ 0.68 & 0 & 0.72 & 0 & 0.72 & 0 \\ 0 & -1 & 0 & -2 & 0 & 3 \\ -1 & -3 & -2 & 4 & 3 & -1 \end{bmatrix}$$

element strains, $\{e\} = [B]\{\delta\}$

$$\{e\} = \begin{Bmatrix} e_r \\ e_\theta \\ e_z \\ \gamma_{rz} \end{Bmatrix} ; \quad \{\delta\} = \begin{Bmatrix} u_1 \\ w_1 \\ u_2 \\ w_2 \\ u_3 \\ w_3 \end{Bmatrix}$$

$$\begin{Bmatrix} e_r \\ e_\theta \\ e_z \\ \gamma_{rz} \end{Bmatrix} = \frac{1}{10}\begin{bmatrix} -3 & 0 & 4 & 0 & -1 & 0 \\ 0.68 & 0 & 0.72 & 0 & 0.72 & 0 \\ 0 & -1 & 0 & -2 & 0 & 3 \\ -1 & -3 & -2 & 4 & 3 & -1 \end{bmatrix}\begin{Bmatrix} 0.002 \\ 0.001 \\ 0.001 \\ -0.004 \\ -0.003 \\ 0.007 \end{Bmatrix}$$

$$= \begin{Bmatrix} 1\times10^{-4} \\ -8\times10^{-6} \\ 28\times10^{-4} \\ -39\times10^{-4} \end{Bmatrix}$$

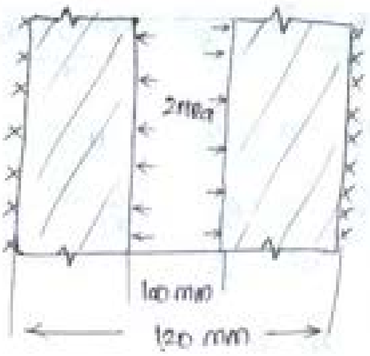
Radial strains, $e_r = 1\times10^{-4}$

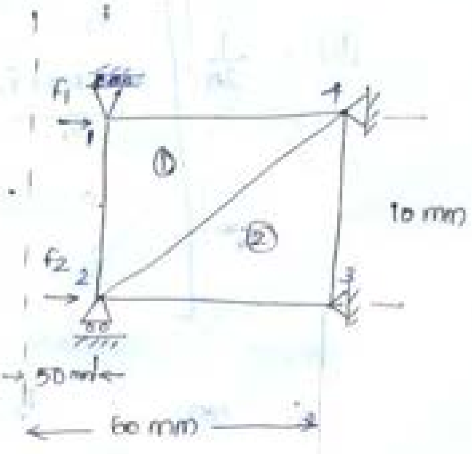
Circumferential strain, $e_\theta = -8\times10^{-6}$

Axial strain, $e_z = 28\times10^{-4}$

Shear strain, $\gamma_{rz} = -39\times10^{-4}$

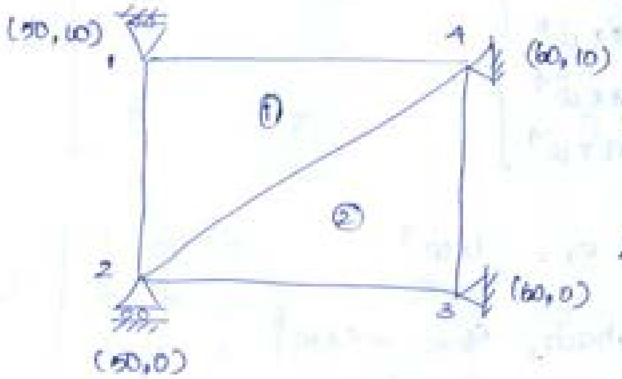
A long hollow cylinder of inside diameter 100 mm and outside diameter 120 mm is firmly fitted in a hole of another rigid cylinder over its full length as shown in fig. The cylinder is then subjected to an internal pressure of 2 MPa. By using two elements on the 10 mm length shown, find the displacements at the inner radius. Take $E = 210$ GPa; $\mu = 0.25$.





Since, the portion contains two elements (1) & (2) of triangular shape as shown in fig.

The global stiffness matrix must be evaluated.



E = 210 GPa

= 2.1×10^5 N/mm²

A = 0.25

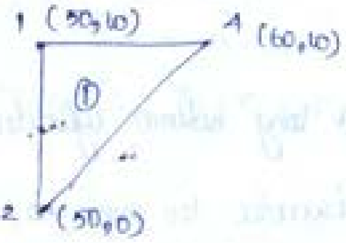
For the element ①:

Nodal coordinates.

$r_1 = 50$ mm ; $z_1 = 10$ mm

$r_2 = 50$ mm ; $Z_2 = 0$

$r_3 = 60$ mm ; $z_3 = 10$ mm



Stiffness matrix, $[K]_1 = 2\pi r A \, [B]^T [D][B]$

$$A = \frac{1}{2} \begin{vmatrix} 1 & 50 & 10 \\ 1 & 50 & 0 \\ 1 & 60 & 10 \end{vmatrix} = 50 \text{ mm}^2 ;$$

$$[B] = \frac{1}{2A} \begin{bmatrix} \beta_1 & 0 & \beta_2 & 0 & \beta_3 & 0 \\ \frac{\alpha_1 + \beta_1 r + \gamma_1 z}{r} & 0 & \frac{\alpha_2 + \beta_2 r + \gamma_2 z}{r} & 0 & \frac{\alpha_3 + \beta_3 r + \gamma_3 z}{r} & 0 \\ 0 & \gamma_1 & 0 & \gamma_2 & 0 & \gamma_3 \\ \gamma_1 & \beta_1 & \gamma_2 & \beta_2 & \gamma_3 & \beta_3 \end{bmatrix}$$

$\alpha_1 = 500$; $\beta_1 = -10$; $\gamma_1 = 10$

$\alpha_2 = 100$; $\beta_2 = 0$; $\gamma_2 = -10$

$\alpha_3 = -500$; $\beta_3 = 10$; $\gamma_3 = 0$

$$r = \frac{r_1 + r_2 + r_3}{3} = \frac{160}{3} ; \qquad z_1 = \frac{z_1 + z_2 + z_3}{3} = \frac{20}{3}$$

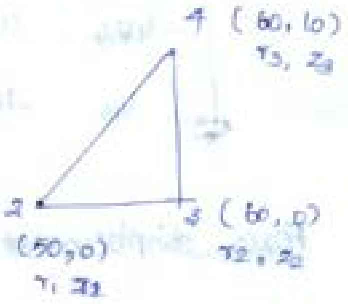
$$[B] = \frac{1}{100} \begin{bmatrix} -10 & 0 & 0 & 0 & 10 & 0 \\ 0.625 & 0 & 0.625 & 0 & 0.625 & 0 \\ 0 & 10 & 0 & -10 & 0 & 0 \\ 10 & -10 & -10 & 0 & 0 & 10 \end{bmatrix}$$

$$[D] = \frac{E}{(1+\mu)(1-2\mu)} \begin{bmatrix} 1-\mu & \mu & \mu & 0 \\ \mu & 1-\mu & \mu & 0 \\ \mu & \mu & 1-\mu & 0 \\ 0 & 0 & 0 & \frac{1-2\mu}{\mu} \end{bmatrix}$$

$$= 24\times10^3 \begin{bmatrix} 3 & 1 & 1 & 0 \\ 1 & 3 & 1 & 0 \\ 1 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$[k]_1 = 10^5 \left[\begin{array}{cccccc|c} u_1 & w_1 & u_2 & w_2 & u_3 & w_3 & \\ 544.1 & -271.3 & -147.1 & 131.3 & -418.4 & 140 & u_1 \\ -271.3 & 560 & 148.8 & -420 & 148.8 & -140 & w_1 \\ -147.1 & 148.8 & 141.7 & -8.75 & 103.9 & -140 & u_2 \\ 131.3 & -420 & -8.75 & 420 & -148.8 & 0 & w_2 \\ -418.4 & 148.8 & 103.9 & -148.8 & 437.1 & 0 & u_3 \\ 140 & -140 & -140 & 0 & 0 & 140 & w_3 \end{array}\right]$$

Similarly for the element ②



$r_1 = 50$; $z_1 = 0$

$r_2 = 60$; $z_2 = 0$

$r_3 = 60$; $z_3 = 10$

$r = \frac{170}{3}$; $z = \frac{10}{3}$;

$$[B] = \frac{1}{100}\begin{bmatrix} -10 & 0 & 10 & 0 & 0 & 0 \\ 0.59 & 0 & 0.59 & 0 & 0.59 & 0 \\ 0 & 0 & 0 & -10 & 0 & 10 \\ 0 & -10 & -10 & 10 & 10 & 0 \end{bmatrix}$$

$$[D] = 24 \times 10^3 \begin{bmatrix} 3 & 1 & 1 & 0 \\ 1 & 3 & 1 & 0 \\ 1 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$[k]_2 = 10^5 \begin{array}{c} \begin{array}{cccccc} u_2 & w_2 & u_3 & w_3 & u_4 & w_4 \end{array} \\ \left[\begin{array}{cccccc} 433.8 & 0 & -448.5 & 141.2 & -7.4 & -141.2 \\ 0 & 150 & 150 & -150 & -150 & 0 \\ -448.5 & 150 & 619.2 & -208.9 & -139.7 & 158.9 \\ 141.2 & -150 & -308.9 & 600 & 141.2 & -450 \\ -7.4 & -150 & -139.7 & 141.2 & 151.5 & 8.9 \\ -141.2 & 0 & 158.9 & -450 & 8.9 & 450 \end{array}\right] \end{array} \begin{array}{c} \\ u_2 \\ w_2 \\ u_3 \\ w_3 \\ u_4 \\ w_4 \end{array}$$

Combining the stiffness matrix $[k]_1$ + $[k]_2$

$$[K] = 10^5 \begin{array}{c} \begin{array}{cccccccc} u_1 & w_1 & u_2 & w_2 & u_3 & w_3 & u_4 & w_4 \end{array} \\ \left[\begin{array}{cccccccc} 544.1 & -271.3 & -147.1 & 131.3 & 0 & 0 & -448.4 & 140 \\ -271.3 & 500 & 148.8 & -420 & 0 & 0 & 148.8 & -40 \\ -147.1 & 148.8 & 576.5 & -8.75 & -448.5 & 141.2 & 96.5 & -281.2 \\ 131.3 & -420 & -8.75 & 570 & 150 & -150 & -298.8 & 0 \\ 0 & 0 & -448.5 & 150 & 619.2 & -308.9 & -139.7 & 158.9 \\ 0 & 0 & 141.2 & -150 & -308.9 & 600 & 141.2 & -450 \\ -448.4 & 148.8 & 96.5 & -298.8 & -139.7 & 141.2 & 590.6 & 8.9 \\ 140 & -140 & -281.2 & 0 & 158.9 & -450 & 8.9 & 590 \end{array}\right] \end{array}$$

Force displacement relationship.

$$\{F\} = [K]\{\delta\}$$

Force acting along the radial direction at nodes 1 & 2

$$F_{1r} = F_{2r} = \frac{2\pi r_i l_e P}{2}$$

r_i = Inner radius = 50 mm

l_e = length of finite element = 10 mm

P = Internal pressure = 2 MPa = 2 N/mm^2.

$$F_{1r} = F_{2r} = \frac{2\pi \times 50 \times 10 \times 2}{2} = 3141.6 \text{ N.}$$

F_{1z}, F_{2z}, F_{3z}, F_{4z} are zero; F_{3r} & F_{4r} → Reaction forces.

$$\{F\} = \begin{Bmatrix} F_{1r} \\ F_{1z} \\ F_{2r} \\ F_{2z} \\ F_{3r} \\ F_{3z} \\ F_{4r} \\ F_{4z} \end{Bmatrix} = \begin{Bmatrix} 3141.6 \\ 0 \\ 3141.6 \\ 0 \\ F_{3r} \\ 0 \\ F_{4r} \\ 0 \end{Bmatrix}$$

Neglecting 2nd, 4th, 5th, 6th, 7th & 8th rows & column in the finite element Equations,

$$10^5 \begin{bmatrix} 544.1 & -147.1 \\ -147.1 & 575.5 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} 3141.6 \\ 3141.6 \end{Bmatrix}$$

Solving, we get

$u_1 = 7.8 \times 10^{-5}$ mm

$u_2 = 7.4 \times 10^{5}$ mm.

UNIT - V - Isoparametric Element / Formulation.

Natural co-ordinate systems - Isoparametric elements - Shape functions for iso parametric elements - one & two dimensions - Serendipity elements - Numerical integration and application to Plane stress Problems - Matrix solution techniques - solutions Techniques to Dynamic problems - Introduction to Analysis software.

Depending upon the nature of boundaries of the systems.

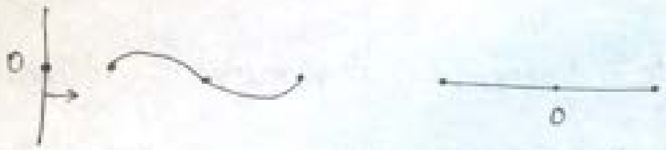
finite element formed as linear (or) Non-linear Elements.
↓ ↓
Straight line   curved Line (or) Side.

Formation of shape functions for non-linear (or) Higher order element in cartesian co-ordinate systems is highly complicated. Inorder to simplify the analysis, a family of elements "Isoparametric elements" are employed.

In Isoparametric elements, the shape functions are derived based on natural co-ordinate system and these shape functions can be utilised to describe both geometric shape and displacements produced due to applied load.

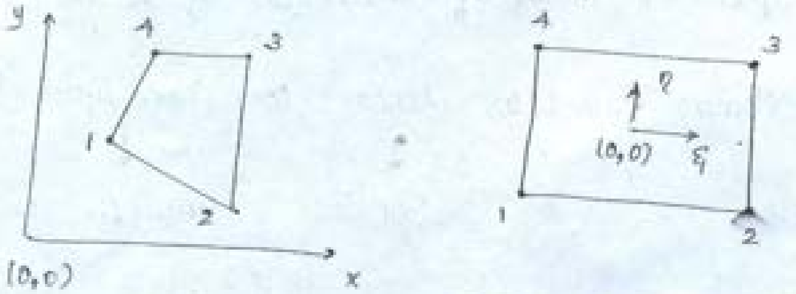
Isoparametric formulation:

Isoparametric elements of simple shapes expressed in natural (local) co-ordinate system which are referred as master - elements.



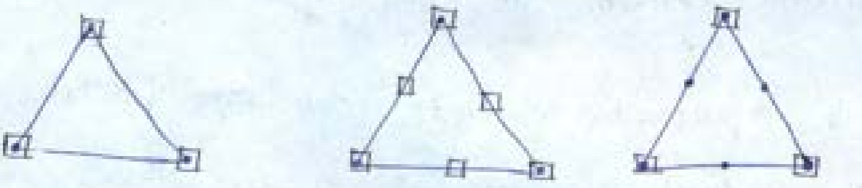
One dimensional line element

Two dimensional triangular element



Two dimensional Quadilateral element

Isoparametric, Super parametric & Sub parametric Elements.



Isoparametric   Super parametric   Sub parametric

• → Nodes for defining displacements

□ → " " " Geometry.

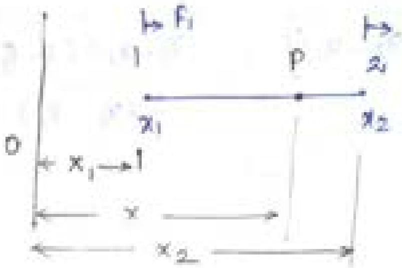
In the Isoparametric element,

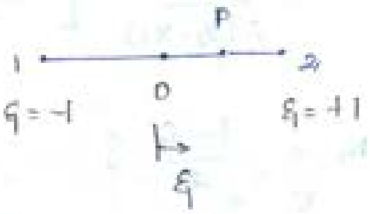
No. of nodes defining geometry & displacements are equal.

i.e, No. of nodes equal to no. of shape functions.

Derivation of shape function for an one dimensional element

Consider a line element in global co-ordinate system shown in fig.





let,

x_1, x_2 = Global co-ordinates of nodes 1 + 2.

u_1, u_2 = Displacements at nodes 1, 2 ; axial load F_1, F_2.

Displacement at the point P with co-ordinate x.

$$u = N_1 u_1 + N_2 u_2$$

Shape functions, $N_1 = \frac{x_2 - x}{x_2 - x_1}$; $N_2 = \frac{x - x_1}{x_2 - x_1}$

Now, consider a natural co-ordinate ξ, whose origin at the centre of the element and its values at node 1 + 2 are -1 and +1.

Value of point P,

$$\frac{\xi + 1}{2} = \frac{x - x_1}{x_2 - x_1} \quad ; \quad \varepsilon = \frac{2(x - x_1)}{x_2 - x_1} - 1 \quad — ①$$

At node 1, $x = x_1$; $\xi = -1$

2, $x = x_2$; $\xi = +1$

Eq(1)

$$x = \frac{(\xi + 1)(x_2 - x_1)}{2} + x_1 \quad —— (2)$$

Substituting x value in the shape function N_1, N_2.

$$N_1 = \frac{x_2 - x}{x_2 - x_1} = \frac{1}{(x_2 - x_1)}\left[x_2 - \left\{ \frac{(\xi+1)(x_2 - x_1)}{2} + x_1 \right\} \right]$$

$$= \frac{1}{2(x_2 - x_1)} \left[2x_2 - \xi x_2 + \xi x_1 - x_2 + x_1 - 2x_1 \right]$$

$$= \frac{1}{2(x_2 - x_1)} \left[(x_2 - x_1) - \xi (x_2 - x_1) \right] = \frac{(x_2 - x_1)(1 - \xi)}{2(x_2 - x_1)}$$

$$N_1 = \frac{1-\xi}{2} ;$$

simillarly, $N_2 = \frac{1+\xi}{2}$;

$$u = N_1 u_1 + N_2 u_2$$

$$= \left(\frac{1-\xi}{2}\right) u_1 + \left(\frac{1+\xi}{2}\right) u_2 \quad \text{—— (3)}$$

Eq(2) →
$$x = \frac{(\xi+1)(x_2 - x_1) + 2x_1}{2}$$

$$= \frac{\xi x_2 - \xi x_1 + x_2 - x_1 + 2x_1}{2}$$

$$x = \left(\frac{1-\xi}{2}\right) x_1 + \left(\frac{1+\xi}{2}\right) x_2 \quad \text{— (4)}$$

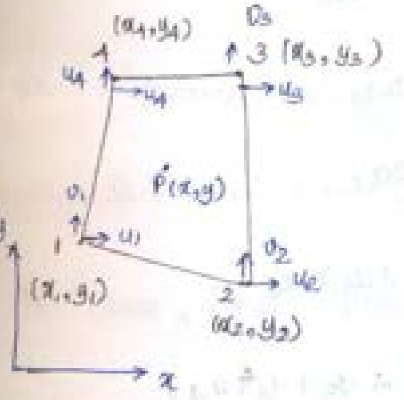
$$x = N_1 x_1 + N_2 x_2.$$

comparing Eq (3) & (4).

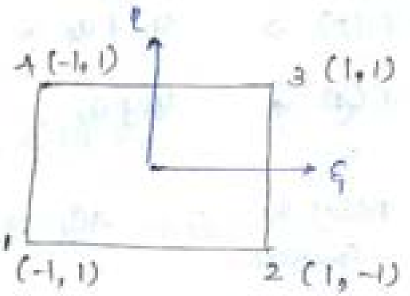
The line element's displacement and geometry are described by the same shape functions.

Derivation of shape functions for a four-noded quadilateral element using natural co-ordinates.

Consider a general quadilateral element specified by global co-ordinate system as shown in fig.



General quadilateral element



Master - element
(Isoparametric quadilateral element).

Let, $\delta_1, \delta_2, \delta_3$ & δ_4 - displacements at node 1, 2, 3 & 4.

u_1, u_2, u_3, u_4 = components along x-axis.

v_1, v_2, v_3, v_4 = " y- axis.

In master element, all the displacements & geometry are specified by natural co-ordinates. ξ & η.

$$u(\xi, \eta) = a_1 + a_2 \xi + a_3 \eta + a_4 \xi \eta$$

$$v(\xi, \eta) = a_5 + a_6 \xi + a_7 \eta + a_8 \xi \eta.$$

a_1 to a_8 → Polynomial co-efficients.

$$u(\xi, \eta) = a_1 + a_2 \xi + a_3 \eta + a_4 \xi \eta.$$

At node 1, $\xi = -1$; $\eta = -1$; and $u = u_1$

2, $\xi = 1$; $\eta = -1$ and $u = u_2$

3, $\xi = 1$; $\eta = 1$ and $u = u_3$

4, $\xi = -1$; $\eta = 1$ and $u = u_4$.

Applying the nodal conditions,

$$u_1 = a_1 - a_2 - a_3 + a_4 \longleftarrow (1)$$

$$u_2 = a_1 + a_2 - a_3 - a_4 \longleftarrow (2)$$

$$u_3 = a_1 + a_2 + a_3 + a_4 \longleftarrow (3)$$

$$u_4 = a_1 - a_2 + a_3 - a_4 . \longleftarrow (4)$$

$(1) + (2) \Rightarrow \quad u_1 + u_2 = 2a_1 - 2a_3$

$(3) + (4) \Rightarrow \quad u_3 + u_4 = 2a_1 + 2a_3 .$

Adding the equations. $\therefore \quad 4a_1 = u_1 + u_2 + u_3 + u_4$

$$a_1 = \frac{1}{4}(u_1 + u_2 + u_3 + u_4).$$

&

Subtracting the eqn. $\quad a_3 = \frac{1}{4}(-u_1 - u_2 + u_3 + u_4)$

Ily

$$a_2 = \frac{1}{4}(-u_1 + u_2 + u_3 - u_4)$$

$$a_4 = \frac{1}{4}(u_1 - u_2 + u_3 - u_4)$$

Substituting a_1 to a_4 in the displacement equations,

$$u = \frac{1}{4}(u_1 + u_2 + u_3 + u_4) + \frac{1}{4}(-u_1 + u_2 + u_3 - u_4)\xi +$$

$$\frac{1}{4}(-u_1 - u_2 + u_3 + u_4)\eta + \frac{1}{4}(u_1 - u_2 + u_3 - u_4)\xi\eta .$$

Rearranging u_1 to u_4, we get

$$u = \frac{1}{4}(1 - \xi - \eta + \xi\eta)u_1 + \frac{1}{4}(1 + \xi - \eta - \xi\eta)u_2 +$$

$$\frac{1}{4}(1 + \xi + \eta + \xi\eta) + \frac{1}{4}(1 - \xi + \eta - \xi\eta)u_4$$

$u = N_1 u_1 + N_2 u_2 + N_3 u_3 + N_4 u_4$.

from which,

$$N_1 = \frac{1}{4}(1-\xi)(1-\eta) \quad ; \quad N_2 = \frac{1}{4}(1+\xi)(1-\eta)$$

$$N_3 = \frac{1}{4}(1+\xi)(1+\eta) \; ; \quad N_4 = \frac{1}{4}(1-\xi)(1+\eta).$$

Illy for displacement v.

$$v = N_1 v_1 + N_2 v_2 + N_3 v_3 + N_4 v_4$$

the same shape function will be obtained.

Nodal displacement at point P,

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_3 \end{Bmatrix}$$

$$\{u\} = [N]\{\delta\}.$$

strain-displacement matrix for a four-noded quadrilateral element using natural co-ordinates.

Strain-displacement relations,

$$\{e\} = \begin{Bmatrix} e_x \\ e_y \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \partial u/\partial x \\ \partial v/\partial y \\ \partial u/\partial y + \frac{\partial v}{\partial x} \end{Bmatrix}$$

displacements,

$$u = N_1 u_1 + N_2 u_2 + N_3 u_3 + N_4 u_4$$

$$v = N_1 v_1 + N_2 v_2 + N_3 v_3 + N_4 v_4$$

Due to isoparametric characteristics. the co-ordinates are also the same.

$$x = N_1 x_1 + N_2 x_2 + N_3 x_3 + N_4 x_4$$

$$y = N_1 y_1 + N_2 y_2 + N_3 y_3 + N_4 y_4$$

Now, the displacement u. can write

$$\frac{\partial u}{\partial \xi} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial \xi} + \frac{\partial u}{\partial y} * \frac{\partial y}{\partial \xi} \quad \text{and}$$

$$\frac{\partial u}{\partial \eta} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial \eta} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial \eta} .$$

In matrix form.

$$\begin{Bmatrix} \frac{\partial u}{\partial \xi} \\ \frac{\partial u}{\partial \eta} \end{Bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \partial y/\partial \xi \\ \frac{\partial x}{\partial \eta} & \partial y/\partial \eta \end{bmatrix} \begin{Bmatrix} \partial u/\partial x \\ \partial u/\partial y \end{Bmatrix}$$

$$= [J] \begin{Bmatrix} \partial u/\partial x \\ \partial u/\partial y \end{Bmatrix} \text{ --- } ①$$

→ named after Carl Jacob Jacobi

[J] is the Jacobian Matrix, which gives relation between the derivatives in the global & Natural co-ordinate system

$$[J] = \begin{bmatrix} \partial x/\partial \xi & \partial y/\partial \xi \\ \partial x/\partial \eta & \partial y/\partial \eta \end{bmatrix} = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix}$$

$$J_{11} = \frac{\partial x}{\partial \xi} = \frac{\partial N_1}{\partial \xi} x_1 + \frac{\partial N_2}{\partial \xi} x_2 + \frac{\partial N_3}{\partial \xi} x_3 + \frac{\partial N_4}{\partial \xi} x_4$$

$$J_{12} = \frac{\partial y}{\partial \xi} = \frac{\partial N_1}{\partial \xi} y_1 + \frac{\partial N_2}{\partial \xi} y_2 + \frac{\partial N_3}{\partial \xi} y_3 + \frac{\partial N_4}{\partial \xi} y_4$$

$$J_{21} = \frac{\partial x}{\partial \eta} = \frac{\partial N_1}{\partial \eta} x_1 + \frac{\partial N_2}{\partial \eta} x_2 + \frac{\partial N_3}{\partial \eta} x_3 + \frac{\partial N_4}{\partial \eta} y_4$$

$$J_{22} = \frac{\partial y}{\partial \eta} = \frac{\partial N_1}{\partial \eta} y_1 + \frac{\partial N_2}{\partial \eta} y_2 + \frac{\partial N_3}{\partial \eta} y_3 + \frac{\partial N_4}{\partial \eta} y_4 .$$

$$N_1 = \frac{1}{4}(1-\xi)(1-\eta) ; \qquad N_2 = \frac{1}{4}(1+\xi)(1-\eta)$$

$$N_3 = \frac{1}{4}(1+\xi)(1+\eta) ; \qquad N_4 = \frac{1}{4}(1-\xi)(1+\eta)$$

$$J_{11} = \frac{1}{4}\left[-(1-\eta)x_1 + (1-\eta)x_2 + (1+\eta)x_3 - (1+\eta)x_4\right]$$

$$J_{12} = \frac{1}{4}\left[-(1-\eta)y_1 + (1-\eta)y_2 + (1+\eta)y_3 - (1+\eta)y_4\right]$$

$$J_{21} = \frac{1}{4}\left[-(1-\xi)x_1 - (1+\xi)x_2 + (1+\xi)x_3 + (1-\xi)x_4\right]$$

$$J_{22} = \frac{1}{4}\left[-(1-\xi)y_1 - (1+\xi)y_2 + (1+\xi)y_3 + (1-\xi)y_4\right]$$

Eq(1) is rewritten into,

$$\begin{Bmatrix} \partial u/\partial x \\ \partial u/\partial y \end{Bmatrix} = [J]^{-1} \begin{Bmatrix} \partial u/\partial \xi \\ \partial u/\partial \eta \end{Bmatrix} \qquad [J]^{-1} = \frac{[\quad]^{T}}{|J|}$$

$$\begin{Bmatrix} \partial u/\partial x \\ \partial u/\partial y \end{Bmatrix} = \frac{1}{|J|} \begin{bmatrix} J_{22} & -J_{12} \\ -J_{21} & J_{11} \end{bmatrix} \begin{Bmatrix} \partial u/\partial \xi \\ \partial u/\partial \eta \end{Bmatrix}$$

Similarly,

$$\begin{Bmatrix} \partial v/\partial x \\ \partial v/\partial y \end{Bmatrix} = \frac{1}{|J|} \begin{bmatrix} J_{22} & -J_{12} \\ -J_{21} & J_{11} \end{bmatrix} \begin{Bmatrix} \partial v/\partial \xi \\ \partial v/\partial \eta \end{Bmatrix}$$

$$\{e\} = \begin{Bmatrix} e_x \\ e_y \\ \gamma_{xy} \end{Bmatrix} = \frac{1}{|J|} \begin{bmatrix} J_{22} & -J_{12} & 0 & 0 \\ 0 & 0 & -J_{21} & J_{11} \\ -J_{21} & J_{11} & J_{22} & -J_{12} \end{bmatrix} \begin{Bmatrix} \partial u/\partial \xi \\ \partial u/\partial \eta \\ \partial v/\partial \xi \\ \partial v/\partial \eta \end{Bmatrix}$$

WKT, displacements,

$$u = N_1 u_1 + N_2 u_2 + N_3 u_3 + N_4 u_4$$

$$v = N_1 v_1 + N_2 v_2 + N_3 v_3 + N_4 v_4$$

$$\frac{\partial u}{\partial \xi} = \frac{\partial N_1}{\partial \xi} u_1 + \frac{\partial N_2}{\partial \xi} u_2 + \frac{\partial N_3}{\partial \xi} u_3 + \frac{\partial N_4}{\partial \xi} u_4$$

$$\partial u/\partial \eta = \frac{\partial N_1}{\partial \eta} u_1 + \frac{\partial N_2}{\partial \xi} u_2 + \frac{\partial N_3}{\partial \eta} u_3 + \frac{\partial N_4}{\partial \eta} u_4$$

$$\partial v/\partial \xi = \frac{\partial N_1}{\partial \xi} v_1 + \frac{\partial N_2}{\partial \xi} v_2 + \frac{\partial N_3}{\partial \xi} v_3 + \frac{\partial N_4}{\partial \xi} v_4$$

$$\partial v/\partial \eta = \frac{\partial N_1}{\partial \eta} v_1 + \frac{\partial N_2}{\partial \eta} v_2 + \frac{\partial N_3}{\partial \eta} v_3 + \frac{\partial N_4}{\partial \eta} v_4$$

Applying the same relation,

$$\frac{\partial u}{\partial \xi} = \frac{1}{4}\left[-(1-\eta)u_1 + (1-\eta)u_2 + (1+\eta)u_3 - (1+\eta)u_4\right]$$

$$\frac{\partial u}{\partial \eta} = \frac{1}{4}\left[-(1-\xi)u_1 + (1+\xi)u_2 + (1+\xi)u_3 + (1-\xi)u_4\right]$$

$$\frac{\partial v}{\partial \xi} = \frac{1}{4}\left[-(1-\eta)v_1 + (1-\eta)v_2 + (1+\eta)v_3 - (1+\eta)v_4\right]$$

$$\frac{\partial v}{\partial \eta} = \frac{1}{4}\left[-(1-\xi)v_1 - (1+\xi)v_2 + (1+\xi)v_3 + (1-\xi)v_4\right]$$

In matrix form,

$$\begin{Bmatrix} \partial u/\partial \xi \\ \partial u/\partial \eta \\ \partial v/\partial \xi \\ \partial v/\partial \eta \end{Bmatrix} = \frac{1}{4}\begin{bmatrix} -(1-\eta) & 0 & (1-\eta) & 0 & 1+\eta & 0 & -(1+\eta) & 0 \\ -(1-\xi) & 0 & -(1+\xi) & 0 & 1+\xi & 0 & 1-\xi & 0 \\ 0 & -(1-\eta) & 0 & 1-\eta & 0 & 1+\eta & 0 & -(1+\eta) \\ 0 & -(1-\xi) & 0 & -(1+\xi) & 0 & 1+\xi & 0 & 1-\xi \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix}$$

Substituting in the strain equations,

$$\begin{Bmatrix} e_x \\ e_y \\ \gamma_{xy} \end{Bmatrix} = \frac{1}{|J|}\begin{bmatrix} J_{22} & -J_{12} & 0 & 0 \\ 0 & 0 & -J_{21} & J_{11} \\ -J_{21} & J_{11} & J_{22} & -J_{12} \end{bmatrix} \times$$

$$\frac{1}{4}\begin{bmatrix} -(1-\eta) & 0 & 1-\eta & 0 & 1+\eta & 0 & -(1+\eta) & 0 \\ -(1-\xi) & 0 & -(1+\xi) & 0 & 1+\xi & 0 & 1-\xi & 0 \\ 0 & -(1-\eta) & 0 & 1-\eta & 0 & 1+\eta & 0 & -(1+\eta) \\ 0 & -(1-\xi) & 0 & -(1+\xi) & 0 & 1+\xi & 0 & 1-\xi \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix}$$

$\{e\} = [B]\{\delta\} = [G][H]\{\delta\}$;

strain-displacement matrix, $[B] = [G][H]$.

$[G]$; $[H]$, $\{\delta\} \rightarrow$ Nodal displacement vector.

stress-strain relationship matrix,

Element stresses, $\{\sigma\} = [D]\{e\} = [D][B]\{\delta\}$

stress-strain relationship / constitutive matrix, $[D]$.

for plane-stress condition,

$$[D] = \frac{E}{(1-\mu^2)} \begin{bmatrix} 1 & \mu & 0 \\ \mu & 1 & 0 \\ 0 & 0 & \frac{1-\mu}{2} \end{bmatrix}$$

for plane-strain condition,

$$[D] = \frac{E}{(1+\mu)(1-2\mu)} \begin{bmatrix} 1-\mu & \mu & 0 \\ \mu & 1-\mu & 0 \\ 0 & 0 & \frac{1-2\mu}{2} \end{bmatrix}$$

stiffness matrix $[K]$ for isoparametric Quadrilateral element.

$$[K] = \int_v [B]^T [D][B]\, dv$$

for isoparametric element

$$[K] = t \iint [B]^T [D][B]\, dx.\underline{dy}$$

$dv = dA \cdot t$
$= dx\, \underline{dy}\, t$

(or)

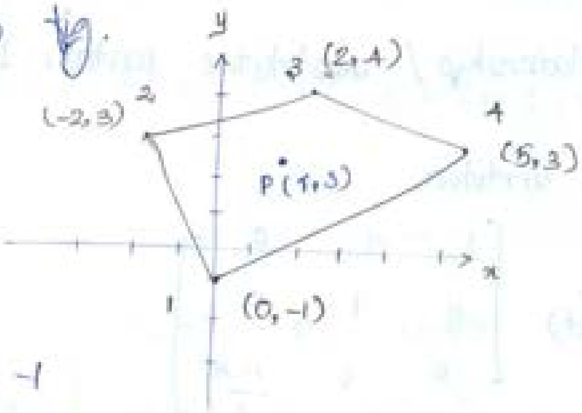
$$[K] = t \int_{-1}^{1}\int_{-1}^{1} [B]^T [D][B]\, \underset{\downarrow}{|J|}\, d\xi\, d\eta.$$

determinant of Jacobian matrix.

The cartesian co-ordinates of the corner nodes of a quadilateral element are given by (0, -1), (-2, 3), (2, 4) and (5, 3). find the co-ordinate transformation between the global and local co-ordinates. Using this, determine the cartesian coordinates of the point defined by (r, s) = (0.5, 0.5) in the local co-ordinate system.

Solution:-

For the given quadilateral element the cartesian coordinates are shown in fig.



$x_1 = 0;\ y_1 = -1$

$x_2 = -2;\ y_2 = 3$

$x_3 = 2;\ y_3 = 4$

$x_4 = 5;\ y_4 = 3.$ and local co-ordinates are,

$r = 0.5;\ s = 0.5.$

The co-ordinate transformation,

$$x = N_1 x_1 + N_2 x_2 + N_3 x_3 + N_4 x_4$$

$$y = N_1 y_1 + N_2 y_2 + N_3 y_3 + N_4 y_4$$

Shape functions are,

$$N_1 = \frac{1}{4}(1-r)(1-s) = \frac{1}{4}(1-0.5)(1-0.5) = 0.0625$$

$$N_2 = \frac{1}{4}(1+r)(1-s) = 0.1875$$

$$N_3 = \frac{1}{4}(1+r)(1+s) = 0.5625$$

$$N_4 = \frac{1}{4}(1-r)(1+s) = 0.1875.$$

Substituting in the Equation,

$$x = (0.0625 \times 0) + (0.1875 \times -2) + (0.5625 \times 2) + (0.1875 \times 5)$$

$$= 1.6875$$

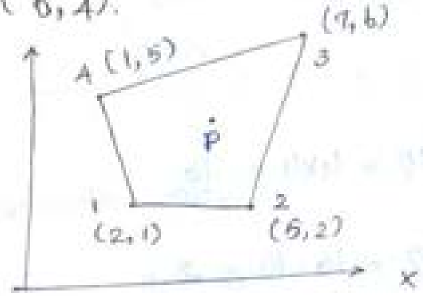
$$y = (0.0625 \times -1) + (0.1875 \times 3) + (0.5625 \times 4) + (0.1875 \times 3)$$

$$= 3.3125.$$

The cartesian co-ordinates of the point P are,

$$(x, y) = (1.6875, 3.3125)$$

For the isoparametric quadrilateral element shown in fig. find the local coordinates of the point P whose cartesian coordinates are (6, 4).



Solution.

cartesian co-ordinates are.

$x_1 = 2$; $y_1 = 1$

$x_2 = 6$; $y_2 = 2$

$x_3 = 7$; $y_3 = 6$

$x_4 = 1$; $y_4 = 5$.

The cartesian co-ordinates of the point $P(x, y) = (6, 4)$

$$x = N_1 x_1 + N_2 x_2 + N_3 x_3 + N_4 x_4$$

$$y = N_1 y_1 + N_1 y_2 + N_3 y_3 + N_4 y_4.$$

The shape functions are,

$$N_1 = \frac{1}{4}(1-\xi)(1-\eta) \; ; \quad N_2 = \frac{1}{4}(1+\xi)(1-\eta)$$

$$N_3 = \frac{1}{4}(1+\xi)(1+\eta) \; ; \quad N_4 = \frac{1}{4}(1-\xi)(1+\eta).$$

Substituting in the equation.

$$x = N_1 x_1 + N_2 x_2 + N_3 x_3 + N_4 x_4$$

$$6 = \frac{1}{4}(1-\xi)(1-\eta)2 + \frac{1}{4}(1+\xi)(1-\eta)5 + \frac{1}{4}(1+\xi)(1+\eta)7 + \frac{1}{4}(1-\xi)(1+\eta)1$$

$$6 = \frac{1}{4}(15 + 9\xi + \eta + 3\xi\eta) \quad (\text{or})$$

$$15 + 9\xi + \eta + 3\xi\eta = 6 \times 4 = 24 \quad \text{— ①.}$$

Ily for y,

$$14 + 2\xi + 8\eta = 4 \times 4 = 16.$$

$$2\xi + 8\eta = 16 - 14 = 2.$$

$$\xi + 4\eta = 1 \; ; \quad \xi = 1 - 4\eta.$$

Substitute in Eq (1).

$$9\xi + \eta + 3\xi\eta = 24 - 15 = 9$$

$$9(1-4\eta) + \eta + 3(1-4\eta)\eta = 9$$

$$-12\eta^2 - 32\eta = 0;$$

$$\eta(12\eta + 32) = 0;$$

$$\eta = 0; \quad 12\eta = -32$$

$$\eta = -32/12 = -2.5.$$

Hence,

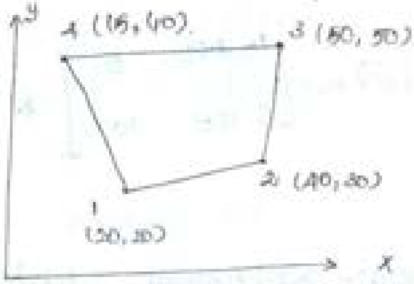
$\xi_1 = 1-4\eta = 1$

$\xi_2 = 1-4\eta = 1-4(-2.6) = 11.68.$

The local co-ordinates of point P are $\xi = 1;\ \eta = 0.$

range from -1 to 1.

In the four-noded quadrilateral element shown in fig. determine the Jacobian and evaluate its value at the point $(\frac{1}{2}, \frac{1}{2})$.



Solution:-

The global co-ordinates are,

At node 1 : $x_1 = 20;\ y_1 = 20$

2 : $x_2 = 40;\ y_2 = 30;$

3 : $x_3 = 50;\ y_3 = 50$

4 : $x_4 = 15;\ y_4 = 40.$

Local co-ordinates are, $\xi = \frac{1}{2} = 0.5;\ \eta = \frac{1}{2} = 0.5.$

Jacobian matrix, $[J] = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix}$

$$J_{11} = \frac{1}{4}(-(1-\eta)x_1 + (1-\eta)x_2 + (1+\eta)x_3 - (1+\eta)x_4)$$

$$J_{12} = \frac{1}{4}(-(1-\eta)y_1 + (1-\eta)y_2 + (1+\eta)y_3 - (1+\eta)y_4)$$

$$J_{21} = \frac{1}{4}(-(1-\xi)x_1 - (1-\xi)x_2 + (1+\xi)x_3 + (1-\xi)x_4)$$

$$J_{22} = \frac{1}{4}(-(1-\xi)y_1 - (1+\xi)y_2 + (1+\xi)y_3 + (1-\xi)y_4)$$

$J_{11} = \frac{1}{4} (-(1-0.5)20 + (1-0.50)40 + (1+0.5)50 - (1+0.5)15]$

$= 15.625$

$J_{12} = 5;$

$J_{21} = 2.125$

$J_{22} = 10.$

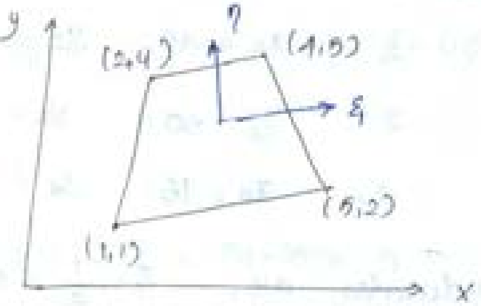
$\therefore [J] = \begin{bmatrix} 15.625 & 5 \\ 2.125 & 10 \end{bmatrix}$

$|J| = \begin{vmatrix} 15.625 & 5 \\ 2.125 & 10 \end{vmatrix} = (15.625 \times 10) - (5 \times 2.125)$

$= 140.625.$

Jacobian matrix $[J] = \begin{bmatrix} 15.625 & 5 \\ 2.125 & 10 \end{bmatrix}$ & its value is 140.625

Establish the strain-displacement matrix for the linear quadrilateral element as shown in fig at Gauss point $(\xi, \eta) = (0.57735, -0.57735)$ in local co-ordinate system.



Solution:

Global co-ordinates are,

$x_1 = 1; \quad y_1 = 1$

$x_2 = 5; \quad y_2 = 2;$

$x_3 = 4; \quad y_3 = 5$

$x_4 = 2; \quad y_4 = 4.$

local co-ordinates of Point P, $\xi = 0.57735; \eta = -0.57735$

Jacobian matrix, $[J] = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix}$

$$J_{11} = \frac{1}{4} \left(-(1-\eta)x_1 + (1-\eta)x_2 + (1+\eta)x_3 - (1+\eta)x_4 \right)$$

$$= \frac{1}{4} \left[-1.57735 + 7.88615 + 1.6906 - 0.8453 \right]$$

$J_{11} = 1.789$

$J_{12} = 0.5$

$J_{21} = -0.289$

$J_{22} = 1.5$

$$[J] = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix} = \begin{bmatrix} 1.789 & 0.5 \\ -0.289 & 1.5 \end{bmatrix}$$

strain-displacement matrix [B]:

$$[B] = [G][H];$$

$$[G] = \frac{1}{|J|} \begin{bmatrix} J_{22} & -J_{12} & 0 & 0 \\ 0 & 0 & -J_{21} & J_{11} \\ -J_{21} & J_{11} & J_{22} & -J_{12} \end{bmatrix}; \quad |J| = \begin{vmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{vmatrix}$$

$$= \frac{1}{2.828} \begin{bmatrix} 1.5 & -0.5 & 0 & 0 \\ 0 & 0 & 0.289 & 1.789 \\ 0.289 & 1.789 & 1.5 & -0.5 \end{bmatrix}$$

$$[H] = \frac{1}{4} \begin{bmatrix} -(1-\eta) & 0 & (1-\eta) & 0 & 1+\eta & 0 & -(1+\eta) & 0 \\ -(1-\xi) & 0 & -(1+\xi) & 0 & (1+\xi) & 0 & 1-\xi & 0 \\ 0 & -(1-\eta) & 0 & 1-\eta & 0 & 1+\eta & 0 & -(1+\eta) \\ 0 & -(1-\xi) & 0 & -(1+\xi) & 0 & 1+\xi & 0 & (1-\xi) \end{bmatrix}$$

$[H] \Rightarrow$

$$[H] = \frac{1}{4}\begin{bmatrix} -1.57735 & 0 & 1.57735 & 0 & 0.42265 & 0 & -0.4226 & 0 \\ -0.42265 & 0 & -1.57735 & 0 & 1.57735 & 0 & 0.4226 & 0 \\ 0 & -1.57735 & 0 & 1.57735 & 0 & 0.42265 & 0 & -0.42 \\ 0 & -0.42265 & 0 & -1.57735 & 0 & 1.57735 & 0 & 0.42 \end{bmatrix}$$

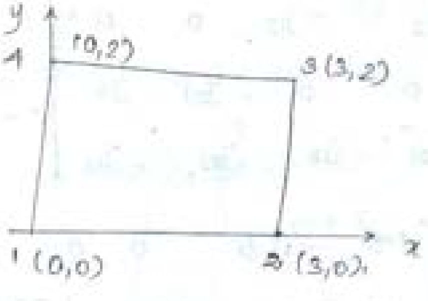
$$\therefore \; [B] = [G][H]$$

$$= \begin{bmatrix} -0.19 & 0 & 0.28 & 0 & -0.01 & 0 & -0.08 & 0 \\ 0 & -0.11 & 0 & -0.21 & 0 & 0.26 & 0 & 0.06 \\ -0.11 & -0.19 & -0.21 & 0.28 & 0.26 & -0.01 & 0.06 & -0.08 \end{bmatrix}$$

For a four noded rectangular element shown in fig. determine the following.

1. Jacobian matrix. 2. Strain - displacement matrix.

3. Element strain 4. Element stresses.



Take, $E = 210\,GPa$; $\mu = 0.25$; $\varepsilon = \eta = 0$;

$$\{\delta\} = [0,\; 0,\; 0.002,\; 0.003,\; 0.005,\; 0.004,\; 0,\; 0]^T$$

Assume plane stress conditions.

Solution:

Global co-ordinates,

$x_1 = 0\ cm$; $y_1 = 0\ cm$

$x_2 = 3\ cm$; $y_2 = 0\ cm$

$x_3 = 3\ cm$; $y_3 = 2\ cm$

$x_4 = 0\ cm$; $y_4 = 2\ cm$

Total find Jacobian matrix [J].

$$[J] = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} -(1-\eta) & 1-\eta & 1+\eta & -(1+\eta) \\ -(1-\xi) & -(1+\xi) & 1+\xi & (1-\xi) \end{bmatrix} \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ x_4 & y_4 \end{bmatrix}$$

substituting the nodal values & ξ, η values,

$$= \frac{1}{4}\begin{bmatrix} -1 & 1 & 1 & -1 \\ -1 & -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 2 & 0 \\ 3 & 2 \\ 0 & 2 \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 6 & 0 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 1.5 & 0 \\ 0 & 1 \end{bmatrix}$$

$$[J] = \begin{bmatrix} 1.5 & 0 \\ 0 & 1.0 \end{bmatrix}$$

Strain - displacement matrix [B]:

$$[B] = [G]\,[H]; \qquad |J| = \begin{vmatrix} 1.5 & 0 \\ 0 & 1 \end{vmatrix} = 1.5$$

$$[G] = \frac{1}{1.5}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1.5 \\ 0 & 1.5 & 1 & 0 \end{bmatrix}$$

$$[H] = \frac{1}{4}\begin{bmatrix} -1 & 0 & 1 & 0 & 1 & 0 & -1 & 0 \\ -1 & 0 & -1 & 0 & 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & +1 & 0 & 1 & 0 & -1 \\ 0 & -1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

$$\therefore [B] = \frac{1}{12}\begin{bmatrix} -2 & 0 & 2 & 0 & 2 & 0 & -2 & 0 \\ 0 & -3 & 0 & -3 & 0 & 3 & 0 & 3 \\ -3 & -2 & -3 & 2 & 3 & 2 & 3 & -2 \end{bmatrix}$$

3. To find element strains $\{e\}$:

$$\{e\} = \begin{Bmatrix} e_x \\ e_y \\ \gamma_{xy} \end{Bmatrix} = [B]\{\delta\}$$

$$= \frac{1}{12} \begin{bmatrix} -2 & 0 & 2 & 0 & 2 & 0 & -2 & 0 \\ 0 & -3 & 0 & -3 & 0 & 3 & 0 & 3 \\ -3 & -2 & -3 & 2 & 3 & 2 & 3 & -2 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 0.002 \\ 0.003 \\ 0.005 \\ 0.004 \\ 0 \\ 0 \end{Bmatrix}$$

$$\{e\} = \frac{1}{12} \begin{bmatrix} 0.014 \\ 0.003 \\ 0.023 \end{bmatrix}$$

4. To find element stresses: $\{\sigma\}$

$$\{\sigma\} = \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = [D]\{e\}$$

$$[D] = \frac{E}{(1-\mu^2)} \begin{bmatrix} 1 & \mu & 0 \\ \mu & 1 & 0 \\ 0 & 0 & \frac{1-\mu}{2} \end{bmatrix}$$ Plane stress condition.

$$= \frac{21 \times 10^6}{(1-0.25^2)} \begin{bmatrix} 1 & 0.25 & 0 \\ 0.25 & 1 & 0 \\ 0 & 0 & \frac{0.75}{3} \end{bmatrix}$$

$$[D] = 56 \times 10^5 \begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 1.5 \end{bmatrix}$$

$$\{\sigma\} = 56 \times 10^5 \begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 1.5 \end{bmatrix} \frac{1}{12} \begin{bmatrix} 0.014 \\ 0.003 \\ 0.023 \end{bmatrix}$$

$$\{\sigma\} = \begin{Bmatrix} 27533 \\ 121,33 \\ 16100 \end{Bmatrix} N/cm^2$$

Numerical integration (Gauss Quadrature method).

Gauss - Legendre quadrature or Gauss quadrature is the approximation method for the definite integrals.

It includes, specific functions called weight functions sampling points called Gauss - points.

consider an integral, $I = \int_a^b f(x)\, dx$.

the value of integral can be approximated as,

$$I = \int_a^b f(x)\, dx = \sum_{i=1}^{n} w_i f(x_i).$$

x_i → sampling points (or) Gauss points.

$f(x_i)$ → values of functions at Gauss points i (1 to n).

It can be expanded as,

$$I = \int_a^b f(x)\, dx = W_1 f(x_1) + W_2 f(x_2) + \dots W_n f(x_n).$$

x → General symbol for any function.

$$I = \int_{-1}^{1} f(\xi)\, d\xi = W_1 f(\xi_1) + W_2 f(\xi_2) + \dots$$

ξ → symbol for natural co-ordinate system.

$f(x)$ is the polynomial of the order

$$f(x) = a_1 + a_2 x$$

$$f(x) = a_1 + a_2 x + a_3 x^2$$

$$f(x) = a_1 + a_2 x + a_3 x^2 + a_4 x^3 \quad \text{for } n = 2.$$

No. of Gauss - points n, equate $(2n-1)$ with the degree of integral function.

Gauss points and weights.

No. of Gauss points (n)	Location of x_i $i = 1$ to n	weight (w_i) $i = 1$ to n.
1	$x_1 = 0$	$w_1 = 2.0$
2	$x_1, x_2 = \pm \frac{1}{\sqrt{3}}$	$W_1 = W_2 = 1.0$
3	$x_1, x_3 = \pm \sqrt{\frac{3}{5}}$ $x_2 = 0.0$	$W_1 = W_3 = \frac{5}{9}$ $W_2 = \frac{8}{9}$
4	$x_1, x_4 = \pm 0.861136$ $x_2, x_3 = \pm 0.339981$	$W_1, w_4 = 0.347855$ $w_2, w_3 = 0.652145$
5	$x_1, x_5 = \pm 0.906180$ $x_2, x_4 = \pm 0.538469$ $x_3 = 0.0$	$w_1, w_5 = 0.236927$ $w_2, w_4 = 0.478629$ $w_3 = 0.56889$

Find the integral $I = \int_{-1}^{1} (2x^3 + 5x^2 + 6)\, dx$ using Gauss quadrature method with 2 point scheme. The Gauss points are ± 0.5774 and the weights at the two points are equal to unity.

Solution:-

As per Gaussian quadrature method,

$$\int_{-1}^{1} f(x)\, dx = W_1 f(x_1) + W_2 f(x_2) \quad \text{for two point scheme.}$$

$$f(x) = 2x^3 + 5x^2 + 6 \quad (\text{Given}).$$

$$x_1 = +0.5774 ; \quad x_2 = -0.5774 \quad (\text{Gauss points})$$

$$w_1 f(x_1) = w_1 \left(2x_1^3 + 5x_1^2 + 6 \right)$$

$$= 1 \left[2(0.5774)^3 + 5(0.5774)^2 + 6 \right]$$

$$= 8.052$$

$$w_2 f(x_2) = w_2 \left(2x_2^3 + 5x_2^2 + 6 \right)$$

$$= 1 \left[2(-0.5774)^3 + 5(-0.5774)^2 + 6 \right]$$

$$= 7.282.$$

Hence,

$$w_1 f(x_1) + w_2 f(x_2) = 8.052 + 7.282 = 15.334.$$

value of integral,

$$I = \int_{-1}^{1} (2x^3 + 5x^2 + 6)\, dx = 15.334.$$

Checking :- with exact value of integral.

$$I = \int_{-1}^{1} (2x^3 + 5x^2 + 6)\, dx$$

$$= 2\left(\frac{x^4}{4}\right)_{-1}^{1} + 5\left(\frac{x^3}{3}\right)_{-1}^{1} + (6x)_{-1}^{1}$$

$$= \frac{1}{2}(1-1) + \frac{5}{3}(1+1) + 6(1+1)$$

$$I = 15.333$$

The solution is equal to the exact value.

Evaluate the integral $\int_{-1}^{1} (x^4+3x^3-x)\,dx$.

Solution:

$$\int_{-1}^{1} f(x)\,dx = \sum_{i=1}^{n} w_i f(x_i).$$

To find, the no. of Gauss - point n, equate $(2n-1)$ with the degree of integral function.

$$2n-1 = 4 \qquad (\text{bcz, } x^4)$$

$$n = \frac{4+1}{2} = 2.5 \simeq 3 \text{ point.}$$

$$\int_{-1}^{1} f(x)\,dx = w_1 f(x_1) + w_2 f(x_2) + w_3 f(x_3)$$

"3 point Gauss function"

$$f(x) = x^4 + 3x^3 - x$$

$$w_1 = \frac{5}{9}; \qquad x_1 = \pm\sqrt{3/5}$$

$$w_2 = \frac{8}{9}; \qquad x_2 = 0.$$

$$w_3 = \frac{5}{9}; \qquad x_3 = -\sqrt{3/5}.$$

$$w_1 f(x_1) = w_1 (x_1^4 + 3x_1^3 + x_1)$$

$$= \frac{5}{9}\left[0.774599^4 + 3(0.774597)^3 - (0.774597)\right]$$

$$= 0.5443.$$

$$w_2 f(x_2) = w_2 (x_2^4 + 3x_2^3 - x_2)$$

$$= 0.88888 \; (0+0+0) = 0.$$

$$w_3 f(x_3) = w_3 (x_3^4 + 3x_3^3 - x_3)$$

$$= 0.5555 \left((-0.774597)^4 + 3(-0.774597)^3 - (-0.774597) \right.$$

$$= -0.1443.$$

$$\int_{-1}^{1}(x^4+3x^3-x)\,dx = 0.5443+0-0.1443 = 0.4.$$

checking:-

$$\int_{-1}^{1}(x^4+3x^3-x)\,dx = \left[\frac{1}{5}x^5+\frac{3}{4}x^4-\frac{1}{2}x^2\right]_{-1}^{1}$$

$$= \frac{2}{5} = 0.4.$$

—

Evaluate the integral $I = \int_{-1}^{1}\left[3e^x+x^2+\frac{1}{(x+2)}\right]dx$ using one point and two-point Gauss-quadrature. compare this with exact solution.

Solution:

$$\int_{-1}^{1} f(x)\,dx = \sum_{i=1}^{n} w_i f(x_i).$$

a) using one-point scheme:-

$$\int_{-1}^{1} f(x)\,dx = W_1 f(x_1)$$

$W_1 = 2.0;$ $x_1 = 0;$

$$w_1 f(x_1) = w_1\left[3e^{x_1}+x_1^2+\frac{1}{(x_1+2)}\right]$$

$$= 2\left[3e^0+0+\frac{1}{0+2}\right]$$

$$= 2(3+0.5) = 7.$$

$$\int_{-1}^{1}\left[3e^x+x^2+\frac{1}{(x+2)}\right]dx = 7.$$

b) Using two-point scheme.

$$\int_{-1}^{1}\left[3e^x+x^2+\frac{1}{(x+2)}\right]dx = W_1 f(x_1)+W_2 f(x_2)$$

$w_1 = w_2 = 1$

$x_1 = \frac{1}{\sqrt{3}}$; $x_2 = -0.57735$

$= 0.57735$;

Now, $w_1 f(x_1) = 1\left[3e^{x_1} + x_1^2 + \frac{1}{(x_1+2)}\right]$

$= 1\left[3e^{0.57735} + (0.57735)^2 + \frac{1}{(0.57735+2)}\right]$

$= [(3 \times 1.7813) + 0.3333 + 0.388]$

$= 6.0652$.

$w_2 f(x_2) = 1\left[3e^{(-0.57735)} + (0.57735)^2 + \frac{1}{(-0.57735+2)}\right]$

$= 2.720$;

$\therefore \int_{-1}^{1}\left[3e^{x} + x^2 + \frac{1}{x+2}\right]dx = 6.0652 + 2.720$

$= 8.7852$.

Exact Solution:-

$\int_{-1}^{1}\left[3e^{x} + x^2 + \frac{1}{(x+2)}\right]dx = \left[3e^{x} + \frac{x^3}{3} + \ln(x+2)\right]_{-1}^{1}$

$= 3\left[e^{1} - e^{-1}\right] + \frac{1}{3}\left[1^3 - (-1)^3\right] +$

$\left[\ln 3 - \ln 1\right]$

$= 8.8158$.

Solution will be exact when the guss point is increased.

Evaluate the integral $I = \int_{-1}^{1}\int_{-1}^{1}(2x^2 + 3xy + 4y^2)\,dx\,.\,dy$

Solution:

$$\int_{-1}^{1}\int_{-1}^{1} f(x,y)\,dx\,dy = \sum_{j=1}^{n}\sum_{i=1}^{n} w_j\, w_i\, f(x_i, y_j)$$

$$f(x,y) = 2x^2 + 3xy + 4y^2.$$

To find the no. of gauss point, n

$$2n-1 = 2$$

$$n = \frac{2+1}{2} = 1.5 \simeq 2 \text{ point.}$$

For 2 point scheme,

$$\int_{-1}^{1}\int_{-1}^{1} f(x,y)\,dx\,dy = w_1^2 f(x, y_1) + w_1 w_2 f(x_1, y_2) + w_2 w_1 f(x_2, y_1) + w_2^2 f(x_2, y_2)$$

For 2 points scheme,

$w_1 = w_2 = 1.0$

$(x_1, y_1) = (+0.57735, +0.57735)$

$(x_2, y_2) = (-0.57735, -0.57735)$

$$w_1^2 f(x_1 y_1) = w_1^2 (2x_1^2 + 3x_1 y_1 + 4y_1^2)$$

$$= 1.0^2 \left[2(0.57735)^2 + 3(0.57735)^2 + 4(0.57735)^2\right]$$

$$= 3.0$$

$$w_1 w_2 f(x, y_2) = w_1 w_2 (2x_1^2 + 3x_1 y_2 + 4y_2^2)$$

$$= 1.0 \times 1.0 \left[2(0.57735)^2 + 3(0.5735 \times -0.57735) + 4(-0.57735)^2\right]$$

$$= 1.0$$

$$w_2 w_1 f(x_2, y_1) = w_2 w_1 (2x_2^2 + 3x_2 y_1 + 4y_1^2)$$

$$= 1.0$$

$$w_2^2 f(x_2, y_2) = w_2^2 (2x_2^2 + 3x_2 y_2 + 4y_2^2)$$

$$= 3.0.$$

Hence,

$$\int_{-1}^{1}\int_{-1}^{1} (2x^2 + 3xy + 4y^2)\, dx \cdot dy = 3 + 1 + 1 + 3$$

$$= 8.0.$$

Exact solution:

$$\int_{-1}^{1}\int_{-1}^{1} (2x^2 + 3xy + 4y^2)\, dx \cdot dy = \int_{-1}^{1} \left[\frac{2}{3}x^3 + \frac{3}{2}x^2 y + 4y^2 x\right]_{-1}^{1} dy$$

$$= \int_{-1}^{1} \left(\frac{4}{3} + 8y^2\right) dy$$

$$= \left[\frac{4}{3}y + \frac{8y^3}{3}\right]_{-1}^{1} = 8.0.$$